On π-asymmetric Geraghty Type Contractive Mappings in Metric Spaces

M. O. Fatai[1], E. E. Aribike[2], H. S. Adewinbi[3], M. T. Mohammed[1] and K. Rauf[1*]

[1]Department of Mathematics, University of Ilorin, Nigeria.
[2]Department of Mathematical Sciences, Lagos State University of Science and Technology, Ikorodu, Lagos, Nigeria.
[3]Department of Mathematical Sciences, Kent State University, United States.

ABSTRACT: Fixed point theory has become an essential tool in the study of nonlinear problems arising in mathematics, science and engineering. However, many of the existing contractive conditions are restrictive or insufficient for establishing the existence and uniqueness of solutions for certain classes of nonlinear problems. The paper aims to introduce a new class of asymmetric Geraghty-type contractive mappings in metric spaces and establish its theoretical properties together with selected practical applications. In order to achieve this aim, a novel asymmetric Geraghty-type contractive condition is formulated for self-mappings on complete metric spaces. The newly proposed contractive condition is then employed to derive sufficient conditions for the existence and uniqueness of fixed points by constructing an appropriate iterative sequence and analysing its convergence behaviour. The stability of the iterative sequence associated with the proposed contraction is further established under suitable assumptions. The applicability of the developed fixed-point theorem is demonstrated by proving the existence and uniqueness of solutions for a first-order nonlinear differential equation arising from a population growth model and the analysis of a mass-spring-damper system governed with second order- nonlinear differential equations, which has important applications in vehicle suspension systems, structural vibration control and machine design. The proposed asymmetric Geraghty-type contractive mapping guaranteeing unique solutions and stable iterative approximations for nonlinear problems and extends the applicability of fixed-point theory for analyzing practical problems encountered in science and engineering.

KEYWORDS: Asymmetric contraction, Convergence, Differential and Integral equations, Fixed point, Geraghty-type mapping, Stability.

[Received Oct. 25, 2025; Revised June 25, 2026; Accepted June 26, 2026] Print ISSN: 0189-9546 | Online ISSN: 2437-2110

I. INTRODUCTION

Fixed point theory is one of the most active branches of nonlinear analysis and has become an indispensable mathematical tool for investigating the existence and uniqueness of solutions to nonlinear problems. Over the past decades, it has found extensive applications in diverse areas including differential and integral equations, optimization theory, engineering, economics, computer science, mathematical biology, and dynamical systems. The versatility of fixed-point techniques in modelling and solving real-world problems has continued to stimulate the development of new contractive conditions with broader applicability and improved analytical flexibility (Combettes & Pesquet 2021; Gdawiec & Adewinbi 2022).

The celebrated Banach contraction principle, introduced by Banach in 1922 [Banach (1922)], remains the cornerstones of fixed-point theory because of its simplicity and powerful applications. Nevertheless, the classical Banach contraction requires a uniform Lipschitz constant strictly less than one, a condition that is often too restrictive for many nonlinear mappings arising in practical applications. This limitation has motivated researchers to develop more general contractive conditions capable of handling broader classes of nonlinear operators while preserving the existence and uniqueness of fixed points.

Among the most influential extensions is the Geraghty contraction introduced by Geraghty in 1973, which replaces the constant contraction coefficient by a control function. Let S denote the family of all functions $\alpha: [0, \infty) \to [0, 1)$, $\lim_{n\to\infty} \alpha(t_n) = 1 \Rightarrow \lim_{n\to\infty} t_n = 0$.

This modification significantly broadened the scope of Banach's contraction principle and provided a more flexible context for studying nonlinear mappings that do not satisfy classical contraction conditions. Following Geraghty's pioneering work, numerous researchers have proposed various generalizations and extensions of Banach and Geraghty contractions. These include the contributions of Rakotch (1962), Cho et al. (2013), Popescu (2014), Karapınar (2014), Faraji (2020), Umudu et al. (2020), Wahab (2023), and

*Corresponding author: krauf@unilorin.edu.ng
doi: http://doi.org/10.63746/njtd.v23i24125

Adewinbi and Andrievskii (2025), among many others. These developments have considerably enriched fixed point theory and expanded its applications to nonlinear differential equations, integral equations, variational inequalities, optimization problems and mathematical models arising in science and engineering.

Despite these significant advances, many existing Geraghty-type contractive conditions are formulated within symmetric metric domain and employ contraction conditions that may not adequately capture asymmetric interactions encountered in several nonlinear models. Consequently, there remains a need for more general contractive conditions that preserve the desirable convergence properties of Geraghty contractions while accommodating asymmetric structures frequently encountered in practical applications. Addressing this limitation is important because asymmetric relationships naturally arise in many physical, biological, and engineering systems where the interaction between variables is direction-dependent and cannot always be represented by conventional symmetric contractions.

Motivated by this observation, this paper introduces a new asymmetric Geraghty-type contractive mapping in metric spaces. The proposed contraction extends several existing Geraghty-type contractive conditions by incorporating an asymmetric control mechanism while maintaining the analytical concept required for fixed point analysis. Using the newly introduced contraction, we establish sufficient conditions for the existence and uniqueness of fixed points for self-mappings on complete metric spaces. Furthermore, we prove the stability of the associated iterative sequence under suitable assumptions. To demonstrate the practical relevance of the theoretical results, the developed fixed-point theorem is applied to establish the existence and uniqueness of solutions of a first-order nonlinear population growth model and to analyse a mass-spring-damper system with second-order nonlinear differential equation, which plays an important role in vehicle suspension systems, structural vibration control, and machine design. These applications illustrate the effectiveness of the proposed contraction and highlight its potential for solving a wider class of nonlinear problems arising in mathematics, science, and engineering.

Unlike the Geraghty contraction of Geraghty (1973), the proposed asymmetric Geraghty-type contraction incorporates an asymmetric control term that allows the contraction condition to depend differently on the arguments of the mapping. Consequently, the proposed concept strictly generalizes several existing Geraghty-type contractions while remaining applicable to nonlinear models that cannot be treated using the classical Banach or Geraghty assumptions.

II. THEORETICAL ANALYSIS

In 2023, Wahab *et al*. established fixed point theorems for ϑ-Quasi-Geraghty Mappings as follows:

Definition 1. *The set $\mathcal{F}^*$ is the class of functions $\alpha_i: \mathbb{R}^+ \to \left[0, \frac{1}{2}\right]$ with the property that $\alpha_i(t_{n,i}) \to \frac{1}{2}$ implies $t_{n,i} \to 0$ for $i = 1,2$.*

Definition 2. *Let Φ be the class of functions $\vartheta: [0, \infty) \to [0, \infty)$ with the property that*

($\vartheta 1$) *ϑ is lower semi-continuous and non-decreasing function;*
($\vartheta 2$) *$\vartheta(t) < t$;*
($\vartheta 3$) *$\vartheta(t) = 0$ if and only if $t = 0$; and*
($\vartheta 4$) *ϑ is subadditive.*

Definition 3. *Let X be a complete metric space and $T: X \to X$ be a self map. The map T is called a ϑ-quasi-Geraghty contractive map if it satisfies*

$$d(Tx, Ty) \leq \alpha\big(d(x,y)\big)\vartheta\big(d(x,y)\big) + \beta\big(d(x,Tx)\big)\vartheta\big(d(x,Tx)\big), \quad (1)$$

for $x, y \in X$, where $\vartheta \in \Phi$ and $\alpha, \beta \in \mathcal{F}^*$.

Definition 4. *Let X be a complete metric space and $T: X \to X$ be a self map. The map T is called quasi-Geraghty contractive map if it satisfies*

$$d(Tx, Ty) \leq \alpha\big(d(x,y)\big)d(x,y) + \beta\big(d(x,Tx)\big)d(x,Tx), \quad (2)$$

for $x, y \in X$, where $\alpha, \beta \in \mathcal{F}^*$.

Lemma 1. *Let $\mathcal{K}$ be a nonempty closed subset of a complete metric space (X, d) and T be a Picard map satisfying* (1) *for which $\vartheta \in \Phi$ and $\alpha, \beta \in \mathcal{F}^*$. Then, for any initial seed x_0, the sequence $\{x_n\}$ given by the Picard map T has a unique fixed point.*

Lemma 2. *Let (K, d) be an arbitrary closed subset of X and T is an operator satisfying contractive condition* (1) *with the property that $\alpha, \beta \in \mathcal{F}^*$ and $F(T)$ is nonempty. Then, for $x_0 \in K$, the sequence $\{x_n\}$ defined by Picard operator is stable.*

Lemma 3 (Miculescu *et al.*). *Let (X, d) be a metric space with $s \geq 1$ and $\{x_n\}$ a sequence in X such that $d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n)$, $n = 1, 2, \cdots$, where $0 \leq r < 1$. Then $\{x_n\}$ is a Cauchy sequence.*

Let $\mathcal{G}^*$ be a family of all functions $\beta: [0, \infty) \to [0, \frac{1}{2})$ that satisfies the condition.

$$\lim_{n\to\infty} \alpha(t_n) = \frac{1}{2} \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Definition 5. *Let Φ be the class of functions $\pi: [0, \infty) \to [0, \infty)$ with the following properties that*

1. *π is lower semi-continuous and non-decreasing function;*
2. *$\pi(t) < t$;*
3. *$\pi(t) = 0$ if and only if $t = 0$ and*
4. *π is subadditive $\pi(a + b) \leq \pi(a) + \pi(b)$*

Give examples of function in the class Φ.

Definition 6. *Let (X, d) be a metric space. A mapping $H: X \to X$ is called a π- asymmetric Geraghty contractive map if it satisfies*

$$d(Tx, Ty) \leq \alpha\big(d(x,y)\big)\pi\big(d(x,y)\big) + \beta\big(d(y,Tx)\big)\pi\big(d(y,Tx)\big) \qquad (3)$$

for all $x, y \in X$, $\pi \in \Phi$ and $\alpha, \beta \in \mathcal{G}^*$.

Definition 7. *Let (X, d) be a metric space. A mapping $H: X \to X$ is called a quasi- asymmetric Geraghty contractive map if it satisfies*

$$\begin{aligned} &d(Tx, Ty) \\ &\leq \alpha\big(d(x,y)\big)\pi\big(d(x,y)\big) \\ &+ \beta\big(d(y,Tx)\big)\pi\big(d(y,Tx)\big) \qquad (4) \end{aligned}$$

for all $x, y \in X$, $\alpha, \beta \in \mathcal{G}^*$, $\pi \in \Phi$ and $\pi(t) = t$.

Theorem 1 (Geraghty (1973)). *Let (X, d) be a complete metric space and let T be a self-mapping on X. Suppose that there exists $\alpha \in S$ such that, for all $x, y \in X$, $d(Tx, Ty) \leq \alpha\big(d(x,y)\big)d(x,y)$, then T has a unique fixed point $x^* \in X$ and $\{T^n x\}$ converges to x^* for all $x \in X$.*

Theorem 2. *Let (X, d) be a complete metric space. Suppose $T: X \to X$ is a π- asymmetric Geraghty contractive mapping. Then T has a unique fixed point.*

Proof

Let $x_0 \in X$ and let x_n be given as $x_{n+1} = Tx_n \;\; n = 0, 1, 2, \cdots$ then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha\big(d(x_{n-1}, x_n)\big)\pi\big(d(x_{n-1}, x_n)\big) + \beta\big(d(x_n, Tx_{n-1})\big)\pi\big(d(x_n, Tx_{n-1})\big) \\ &\leq \alpha\big(d(x_{n-1}, x_n)\big)\pi\big(d(x_{n-1}, x_n)\big) + \beta\big(d(x_n, Tx_{n-1})\big)\pi\big(d(x_n, x_n)\big). \end{aligned}$$

Therefore,

$$d(x_n, x_{n+1}) \leq \alpha\big(d(x_{n-1}, x_n)\big)\pi\big(d(x_{n-1}, x_n)\big) \qquad (5)$$

since $\alpha \in \mathcal{G}^*$ then

$$d(x_n, x_{n+1}) \leq \pi\big(d(x_{n-1}, x_n)\big)$$

since $\pi \in \Phi$, then

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$$

this implies that, $d(x_n, x_{n+1})$ is non increasing, nonnegative term and thus converges to a nonnegative real number δ such that

$$\delta = \liminf_{n\to\infty} d(x_n, x_{n+1}) = \liminf_{n\to\infty} d(x_{n+1}, x_n) \;\; as \;\; n \to 0$$

$$\delta \leq \alpha(\delta)\pi(\delta) < \frac{1}{2}\delta$$

and this shows contradiction.

Hence,

$$\lim_{n\to\infty} inf\, d(x_n, x_{n+1}) = 0$$

Next is to prove that $\{x_n\}$ is Cauchy sequence. From Equation [5], we have

$$d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n), \qquad (6)$$

where $r < 1$ since $\alpha \in \mathcal{G}^*$. By Lemma [3], $\{x_n\}$ is Cauchy sequence. By completeness of X, x_n converges to $x_* \in X$. Now, we need to prove that $Tx_* \in X$, that is $Tx^* = x^*$. Then,

$$\begin{aligned} d(Tx_*, x_*) &\leq d(Tx_*, x_{n+1}) + d(x_{n+1}, x_*) \\ &= d(Tx_*, Tx_n) + d(x_{n+1}, x_*) \\ &\leq \alpha\big(d(x_*, x_n)\big)\pi\big(d(x_*, x_n)\big) \\ &\quad + \beta(d(x_n, Tx_*)\pi\big(d(x_n, Tx_*)\big) + d(x_{n+1}, x_*) \\ &\leq \alpha\big(d(x_*, x_n)\big)d(x_*, x_n) \\ &\quad + \beta(d(x_n, Tx_*)d(x_n, Tx_*) + d(x_{n+1}, x_*) \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\Big(1 - \beta\big(d(x_*, Tx_*)\big)\Big)\, d(x_*, Tx_*) \leq 0.$$

Since $1 - \beta\big(d(x_*, Tx_*)\big) > 0$,

therefore we have $d(Tx_*, x_*) = 0$ which implies that $x_* = Tx_*$.

Next, assume that x_* and y_* are two fixed point of T with $d(x^*, y^*) \neq 0$. Then,

$$\begin{aligned} d(x_*, y_*) = d(Tx_*, Ty_*) &\leq \alpha\big(d(x_*, y_*)\big)\pi\big(d(x_*, y_*)\big) \\ &\quad + \beta(d(y_*, Tx_*\,))\pi(d(y_*, Txx_*\,)) \\ &= \big[\alpha\big(d(x_*, y_*)\big) + \beta\big(d(x_*, y_*)\big)\big]\pi\big(d(x_*, y_*)\big). \end{aligned}$$

It is by the properties on $\alpha, \beta\; and\; \pi$ which reduces to

$$d(x_*, y_*) \leq d(x*, y*)$$

This is a contradiction and therefore $x_* = y_*$.

Corollary 1. *Let (X, d) be a complete metric space. A mapping $T: X \to X$ be a quasi π- asymmetric Geraghty contractive mapping* (4). *Then T has a unique fixed point.*

Theorem 3. *Let M be a non-empty closed subset of X and T is an operator satisfying the contractive condition* (3) *with the property that $\alpha, \beta \in G$ and $F(T)$ is non-empty. Then, for $n_0 \in M$. The sequence $\{x_n\}$ defined by Picard operator is stable.*

Proof

Let $\{y_n\} \subset M$ be an arbitary sequence and let $\rho_n = d(x_n, Ty_n)$ Let $x^* \in F(T)$ and assume that $\rho_n \to 0 \ \ as \ \ n \to \infty$ then, by the hypothesis

$$d(Ty_n, Tx_*) \leq d(y_n, Ty_n) + d(Ty_n, Tx_*)$$

But

$$d(Ty_n, Tx_*) \quad \leq \alpha\big(d(y_n, x_*)\big)\pi\big(d(y_n, x_*)\big) + \beta\big(d(x{\cdot}Ty_n)\big)\pi\big(d(x{\cdot}Ty_n)\big)$$

Thus,

$$d(y_n, x_*) \leq \rho_n + \alpha\big(d(y_n, x_*)\big)\pi\big(d(y_n, x_*)\big) + \beta(\rho_n)\pi(\rho_n)$$

Since $\rho_n \to 0$ and $\alpha, \beta \ \ and \ \pi$

$$\frac{1}{2}d(y_n, x^*) \leq \rho_n + \frac{1}{2}\pi(\rho_n) \to \ \ as\ n \to \infty$$

Then, $y_n \to x^* \in F(T)$. Suppose $d(y_n, x_*) \to 0$ for large n where $x_* \in F(T)$, then by hypothesis,

$$\begin{aligned}\rho_n \quad &= d(x_n, Ty_n) \leq d(x_n, x_*) + d(Hx{\cdot}Ty_n)\\ &\leq d(x_n, x_*) + \alpha\big(d(x{\cdot}y_n)\big)\sigma\big(d(x_*, y_n)\big) + \beta\big(d(Ty_n, x_*)\big)\sigma\big(d(Ty_n, x_*)\big)\\ &\leq d(x_n, x_*) + \alpha\big(d(x_*, y_n)\big)\sigma\big(d(x_*, y_n)\big)\end{aligned}$$

which implies that

$$\rho_n \leq \quad \frac{1}{2}\big(d(x_*, y_n)\big) \to 0$$
$$\rho_n \to 0 \Longrightarrow y_n \to x_*$$

and $y_n \to x_* \Longrightarrow \rho_n \to 0$

Therefore, the operator satisfying (3) is stable.

III. APPLICATIONS

A. A Case of First Order Non-Linear Ordinary Differential Equation

Several real-life situations can be expressed in form of non-linear ordinary differential equations [Albert (2019)]. Examples of such situations are population growth at time t, say $x(t)$. This can be given in the form of differential equation:

$$\frac{dx}{dt} = f(t, y); \quad x(0) = x_0 \quad (7)$$

The equation (7) can be written in the form of an integral expression:

$$x(t) = x_0 + \int_0^t f(s, x(t))ds \quad (8)$$

where $f(t) \in H$ and $f: I \times I \to \mathbb{R}$ is a positive function.
Let V be a class of function spaces which is stronger than $(V(I, \mathbb{R}), d)$ and has the property that $x(t) \in V \Leftrightarrow \int_0^{\mathrm{t}} f(s, x(s))\, ds \ \in\ V$ whenever $f(t) \ \in\ V$.

This condition leads to the admissibility of the pair (V, V) with respect to equation (8).

Let $G = V(I, \mathbb{R})$ be the set of all continuous real-valued functions defined on $I = [0, t]$ and

$$d : G \times G \to \mathbb{R}^+ \ be\ defined\ by$$
$$d(x, y) = sup\_\{t \in I\}\, |x(t) - y(t)| \quad for\, x, y \in V(I, \mathbb{R})$$

Clearly the space $(V(I, \mathbb{R}), d)$ is a complete metric space.

The following hypotheses are imposed on V for its admissibility.

1. The pair (V, V) is admissible with respect to (a8).
2. For each $x(t) \in V$, there corresponds $x(t) \in Tx(t)$ such that $Tx(t) \in V$ for $t \in I$.
3. There exist positive functions ϕ_1 and ϕ_2 such that:

$$\int_0^t [|f(s, x) - f(s, y)|]\, ds \leq \phi_1(x - y) + \phi_2(y - Hx)$$

4. The function $\phi_1(t) = 1 - e^{-kt}$ and $\phi_2(s) = ln(1 + ks)$.
5. $\phi(t) < t$ for each $t > 0$.

With respect to the above definition and conditions, result concerning equation (8) is presented as follows:

Theorem 4. Suppose that all hypotheses 1 to 5 are satisfied by (8), then there exists a unique solution of equation (8) belonging to V.

Proof. Let $H : x(t) \to x(t)$ from V to V where $x(t)$ is the solution of (8) and $x(t)$ is such that

$$x(t) \in x_0 + \int_0^t f(s, x(s))\, ds$$

Let $H : y(t) \to y(t)$ be such that

$$y(t) \in x_0 + \int_0^t f\big(s, y(s)\big)ds$$

Going by the condition (1) to (2), it follows that

$$\begin{aligned} d(Hx, Hy) = \ & d(x,y) \\ & = \left| x_0 + \int_0^t f\big(s, x(s)\big)ds - x_0 - \int_0^t f\big(s, y(s)\big)ds \right| \\ & = \left| \int_0^t f\big(s, x(s)\big)ds - \int_0^t f\big(s, y(s)\big)ds \right| \\ & \leq \int_0^t \big[\big|f\big(s, x(s)\big) - f\big(s, y(s)\big)\big|\big]\ ds \end{aligned}$$

$$\leq \phi_1(x - y) + \phi_2(y - x)$$

$$\leq \frac{1 - e^{-\frac{1}{2}|x-y|}}{L} + \frac{ln(1 + \frac{1}{2}|y - x|)}{L}$$

$$= \frac{1 - e^{-\frac{1}{2}|x-y|}}{L|x - y|} \cdot sup\,\{|x - y|\} + \frac{ln(1 + \frac{1}{2}|y - x|)}{L|y - x|} \cdot sup\,\{|y - x|\}$$

Since $x \in H_x$, we have

$$d(Hx, Hy) \leq \frac{1 - e^{-\frac{1}{2}|x-y|}}{L|x - y|} \cdot \sup\{|x - y|\} + \frac{ln(1 + \frac{1}{2}|y - x|)}{L|y - x|} \cdot \sup\{|y - x|\}$$

$$d(Hx, Hy) \leq \frac{1 - e^{-\frac{1}{2}|x-y|}}{L|x - y|} \cdot \sup\{|x - y|\} + \frac{ln(1 + \frac{1}{2}|y - Hx|)}{L|y - Hx|} \cdot \sup\{|y - Hx|\}$$

$$= \alpha(d(x,y))\,\pi(d(x,y)) + \beta(d(y,Hx))\,\pi(d(y,Hx))$$

Therefore, $x(t)$ satisfies the new Geraghty mapping and hence has a unique solution in a class of function space C_h.

B. A Case of Second Order Non-Linear Ordinary Differential Equation

Considering a mass-spring-damper system problem in Physics and Engineering. The model has a wide application in vibration analysis, suspension system and Structural analysis. Such applications are car Suspension, Building design, Machine designs among others. A mass-Spring-damper System problem is in the form of second order non-linear ordinary differential equation given as $x''(t) = f(t, x(t), x'(t))$ where $x''(t)$ is the acceleration, $x'(t)$ is the velocity and x is the position at the time, t. Now, considering the boundary value problem:

$$x''(t) = f\big(t, x(t), x'(t)\big) \qquad (9)$$

The equation above can be written in the form of an integral expression:

$$x(t) = x_0 + \int_0^t G(t,s) f(s, x(s), x'(t))ds \quad (10)$$

where $G(t, s)$ is the green function. In order to prove the existence of the solution to equation (10), we have the following properties:
Let C_h be a class of function spaces which is stronger than $(C(I, \mathbb{R}), d)$ and has the property that $x(t) \in C_h \Leftrightarrow \int_0^t G(t,s) f(s, x(s), x'(s))ds \in C_h$. This condition leads to the admissibility of the pair (C_h, C_h) with respect to equation (8).

Let $G = C(I, \mathbb{R})$ be the set of all continuous real-valued functions defined on $I = [0, t]$ and
$d : G \times G \to \mathbb{R}^+$ be defined by

$$d(x,y) = \sup_{t \in I} |x(t) - y(t)|, \quad for\, x, y \in C(I, \mathbb{R})$$

Clearly the space $(C(I, \mathbb{R}), d)$ is a complete metric space.

The following hypotheses are imposed on C_h for its admissibility.
(i.) The pair (C_h, C_h) is admissible with respect to (10).
(ii.) For each $x(t) \in C_h$, there corresponds $x(t) \in Hx(t)$ such that $Hx(t) \in C_h$ for $t \in I$.
(iii.) There exist positive functions ϕ_1 and ϕ_2 such that:

$$\|f(s, x(s), x'(s)) - f(s, y(s), y'(s))\| \leq \phi_1(x,y) + \phi_2(x', y')$$

and $\phi_1(x, y) \leq \|x - y\|$ and $\phi_2(x', y') \leq |x' - y'|$ $such\ that$ $|x' - y'| \leq |x - y|$
(iv.) The green function $G(t, s)$, is given by

$$G(t,s) = \begin{cases} t(1-s) & for\ 0 \leq t \leq s \leq 1 \\ s(1-t) & for\ 0 \leq s \leq t \leq 1 \end{cases}$$

and $\int_I G(t,s)ds \leq \frac{1}{8}$
With respect to the above definition and conditions, result concerning equation (a10) is presented as follows:
Theorem 5. Suppose that all hypotheses (i.) to (iii.) are satisfied by (10), then there exists a unique solution of equation (10) belonging to C_h.

Proof:
Let $H: x(t) \to x(t)$ from C_h to C_h where $x(t)$ is the solution of (10) and $x(t)$ is such that:

$$x(t) \in \int_0^t G(t,s)[f(s, x(s), x'(s))ds \in C_h$$

Let $H: y(t) \to y(t)$ be such that:

$$y(t) \in \int_0^t G(t,s)[f(s,y(s),y'(s))ds \in C_h$$

Going by the condition 1 to 3, it follows that:

$$d(Hx,Hy) = d(x,y)$$
$$= \left|\int_0^t G(t,s)[f(s,x(s),x'(s))ds - \int_0^t G(t,s)[f(s,y(s),y'(s))ds\right|$$
$$= \left|\int_0^t G(t,s)[f(s,x(s),x'(s)) - f(s,y(s),y'(s))]ds\right|$$
$$\leq \int_0^t G(t,s)ds[|f(s,x(s),x'(s)) - f(s,y(s),y'(s))|]ds$$
$$\leq \left(\frac{1}{8}\right)[\phi_1(x,y) + \phi_2(x',y')]$$
$$\leq \left(\frac{1}{8}\right)[|x-y| + |x-y|]$$
$$= \left(\frac{1}{8}\right)|x-y| + \left(\frac{1}{8}\right)|x-y|$$
$$\leq \frac{|x-y|}{2|x-y|} \cdot \frac{sup|x-y|}{4} + \frac{|y-x|}{2|y-x|} \cdot \frac{sup|y-x|}{4}$$

Since $x \in H_x$, we have

$$d(Hx,Hy) \leq \frac{|x-y|}{2|x-y|} \cdot \frac{sup|x-y|}{4} + \frac{|y-Hx|}{2|y-Hx|} \cdot \frac{sup|y-Hx|}{4}$$
$$= \alpha(d(x,y))\pi(d(x,y)) + \beta(d(y,Hx))\pi(d(y,Hx))$$

Therefore $x(t)$ satisfies the new geraghty mapping and hence has a unique solution in C_h. ■

IV. ESTIMATION

Here, we will consider three numerical implementations to compare the new asymmetric Geraghty-type contraction with other three existing Geraghty contractions. This is to check the estimates of mappings with respect to different type of functions used.

Example 1: Consider a log function $H(x) = x\,log(1+x)$ with $x_0 = 0.5$. The table and graph for the estimate of example 1 are presented below.

Table 1 Estimate for Example 1.

N	ξ_n1	ξ_n2	ξ_n3	ξ_n4	Min estimate
1	1.428571e-01	2.500000e-01	3.333333e-01	1.875000e-01	1.428571e-01
5	9.519843e-04	1.562500e-02	6.584362e-02	3.707886e-03	9.519843e-04
10	1.812548e-06	4.882812e-04	8.670765e-03	2.749683e-05	1.812548e-06
20	6.570662e-12	4.768372e-07	1.503643e-04	1.512152e-09	6.570662e-12
50	3.130169e-28	4.440892e-16	7.841643e-10	2.514973e-22	3.130169e-28
100	1.959592e-55	3.944305e-31	1.229827e-18	1.265018e-43	1.959592e-55
1000	0.000000e+00	4.666318e-302	4.052387e-177	0.000000e+00	0.000000e+00

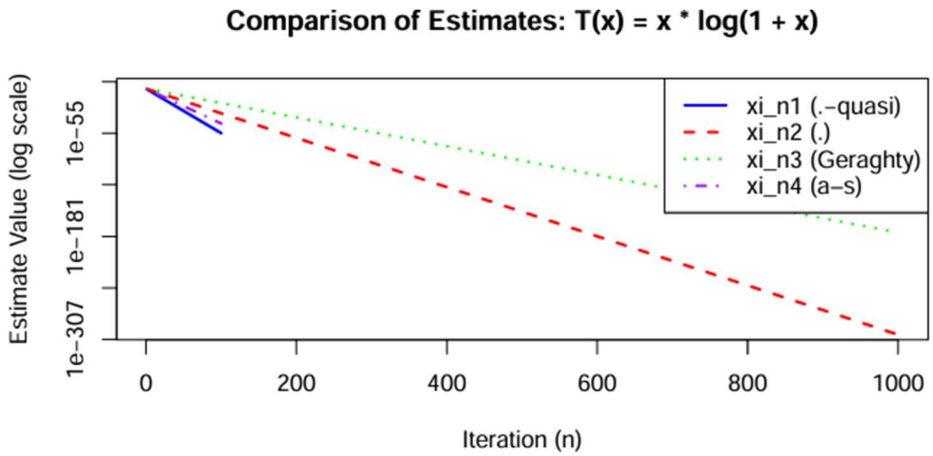

Figure 1

IV. RESULTS AND DISCUSSION

The theoretical framework developed in this paper represents a significant advancement in fixed point theory by introducing an asymmetric Geraghty-type contractive condition that extends the classical Geraghty contraction to accommodate direction-dependent interactions. The proposed contraction, defined in Definition 6, incorporates two control functions α and β acting on different arguments (d(x,y) and d(y,Tx)) thereby capturing asymmetric relationships that naturally arise in many physical and biological systems. This asymmetry distinguishes the presented approach from existing Geraghty-type contractions, which typically employ symmetric conditions that may fail to model direction-dependent phenomena adequately. The proof of Theorem 2 establishes both existence and uniqueness of fixed points under relatively mild assumptions, with the key insight being that the asymmetry does not compromise convergence properties. The stability result in Theorem 3 further demonstrates that the iterative sequence generated by the proposed contraction exhibits desirable robustness properties, which is essential for practical numerical implementations. The numerical examples presented in Section IV provide compelling evidence of the superiority of the asymmetric contraction over classical Geraghty and quasi variants, consistently achieving lower error estimates across all iteration counts. This improved performance is particularly evident for logarithmic functions, where the asymmetric mapping demonstrates enhanced convergence rates and stability.

The applications to population growth models and mass-spring-damper systems validate the practical utility of the theoretical results, demonstrating that the proposed contraction can establish unique solutions for nonlinear differential equations arising in real-world engineering and scientific problems. These applications highlight the broader relevance of asymmetric contractive conditions for modelling complex systems where interactions between variables depend on direction. However, the current framework is limited to self-mappings on complete metric spaces, and extensions to multivalued mappings, partial metric spaces, or more general topological structures remain important directions for future investigation.

V. CONCLUSION

This paper introduced a new class of asymmetric Geraghty-type contractive mappings in complete metric spaces, thereby extending the scope of classical Geraghty contractions to a more general asymmetric framework. By employing the proposed contractive condition, sufficient conditions were established for the existence and uniqueness of fixed points of self-mappings, while the stability of the associated iterative sequence was rigorously proved. The theoretical results were further validated through applications to a first-order nonlinear population growth model and a mass-spring-damper system, demonstrating the effectiveness of the proposed approach in establishing the existence and uniqueness of solutions to nonlinear differential equations arising in science and engineering. The proposed asymmetric Geraghty-type contraction not only broadens the applicability of fixed-point theory but also provides a flexible analytical framework for studying nonlinear problems that may not be adequately addressed by existing contractive conditions. These findings contribute to the ongoing development of generalized contraction principles and reinforce the importance of fixed-point techniques in both theoretical analysis and real-world mathematical modelling. Future research may focus on extending the proposed contraction to multivalued mappings, partial metric spaces, (b)-metric spaces, fuzzy metric spaces, partial differential equations, fractional differential equations, and other nonlinear dynamical systems. Such investigations are expected to further enhance the applicability of the proposed framework and stimulate new developments in fixed point theory and its interdisciplinary applications.

ACKNOWLEDGEMENT

The authors are grateful to Federal University Oye-Ekiti, University of Ilorin and Lagos State University of Science and Technology for the supports they received during the compilation of this work.

AUTHOR CONTRIBUTIONS

M. O. Fatai: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Resources, Writing—original draft and editing. **E. E. Aribike:** Formal analysis,

Funding acquisition, Investigation, Resources and editing. **H. S. Adewinbi:** Investigation, Methodology, Software, Validation, Visualization and editing. **M. T. Mohammed:** conceptualization, Investigation and Validation. **K. Rauf:** conceptualization, Methodology, Project administration, Supervision, Validation, Visualization, writing—review and editing.

Competing interests

The manuscript was read and approved by all the authors. They therefore declare that there is no conflict of interest.

Funding

The Authors received no financial support for the research, authorship, and/or publication of this article.

REFERENCES

Combettes, P. L., & Pesquet, J. C. (2021). Fixed point strategies in data science. *IEEE Transactions on Signal Processing, 69,* 3878-3905.

Gdawiec, K., & Adewinbi, H. (2022). Procedural generation of artistic patterns using a modified orbit trap method. *Applied Sciences, 12*(6), 2923.

Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math., 3*, 133–181.

Geraghty, M. (1973). On contractive mappings. *Proceeding American Mathematical Society, 40*(2), 604–608.

Rakotch, E. (1962). A note on contractive mappings, *Proceeding American Mathematical Society, 13*, 459–465.

Cho, S. H., Bae, J. S., & Karapınar, E. (2013). Fixed point theorems for α-Geraghty contraction type maps in metric spaces. *Fixed point theory and applications, 2013*(1), 329.

Popescu, O. (2014). Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces. *Fixed Point Theory and Applications, 2014*(1), 190.

Karapinar, E. (2014). A Discussion on "α-ψ Geraghty Contraction Type Mappings". *Filomat, 28*(4), 761-766.

Umudu, J. C., Olaleru, J. O., & Mogbademu, A. A. (2020). Fixed point results for Geraghty quasi-contraction type mappings in dislocated quasi-metric spaces. *Fixed Point Theory and Applications, 2020*(1), 16.

Adewinbi, H. S., & Andrievskii, V. (2025). Results on Singh–Chatterjea Type Contractive Mappings in b-Metric Spaces. *Preprints*. https://doi.org/10.20944/preprints202512.2638.v1

Faraji, M. (2020). Fixed points of (α, φ) Geraghty contractions in metric-like spaces. *Fixed Point Theory Applications.* 1-12.

Wahab, O. T. (2023). On ϑ-quasi-Geraghty Contractive Mappings and Application to Perturbed Volterra and Hypergeometric Operators. Kyungpook Mathematical Journal, 63(1), 45-60.

Miculescu, R. and Mihail, A. (2017), New fixed point theorems for set-valued contractions in b-metric spaces, J. Fixed Point Theory Appl. 19 (2017), no. 3, 2153–2163.

Albert, I. (2019). Fixed points and nonlinear PDEs. *J. Differential Equations*, 1-15.