

Approximation of Three-Steps Iterative Schemes in Banach Spaces using Zamfirescu Operator and their Applications



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ABSTRACT: Iterative methods play an important role in fixed point theory and provide effective procedures for approximating fixed points of nonlinear operators. Among the various iterative schemes developed for this purpose, the Noor iteration has attracted considerable attention because of its applicability to a broad class of nonlinear problems. In this study, a modified form of the Noor iterative scheme is introduced with the aim of improving its convergence behavior. The proposed modification is constructed by appropriately altering the Noor iteration while retaining its basic iterative structure. The convergence characteristics of the original Noor iteration and the proposed modified Noor iteration are then investigated and compared. To assess the effectiveness of the proposed scheme, two numerical examples are considered for nonlinear mappings possessing fixed points. The iterative processes are implemented from the same initial approximation and under comparable control parameters, allowing their convergence behaviors to be examined on a common basis. The numerical results demonstrate that the modified Noor iteration approaches the fixed point more rapidly than the original Noor iteration in the examples considered. In particular, the proposed scheme requires fewer iterations to attain a prescribed level of accuracy, indicating an improvement in computational efficiency. These observations provide numerical evidence that the modification enhances the convergence performance of the Noor scheme. The results are therefore useful in the development and analysis of efficient iterative procedures for approximating fixed points of nonlinear operators and related nonlinear problems. The numerical examples used demonstrates faster convergence of the proposed Modified Noor iteration for the nonlinear heat-transfer model considered in this paper.

KEYWORDS: Contraction mapping, Convergent rate, Convex metric Space, Uniformly convex Banach space, convex subset, Fixed point iteration.

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I. INTRODUCTION

Fixed point theory is an important area of nonlinear analysis with applications in differential equations, optimization, integral equations, numerical analysis, and many other branches of mathematics. One of the fundamental approaches for approximating fixed points is the method of successive approximations, which has its origins in the study of existence and approximation of solutions to differential equations. Early contributions are associated with Liouville and Sturm in 1837 and Picard in 1890, while subsequent developments include the iterative processes introduced by Mann (1953) and other researchers. Fixed point theory was formally established as a significant area of mathematical analysis in the early twentieth century, with a major milestone being the celebrated work of Stefan Banach in 1922. Banach's contraction principle not only guarantees the existence and uniqueness of a fixed point under

suitable conditions but also provides a constructive iterative procedure for approximating the fixed point. Iterative procedures play an important role in the numerical solution of equations arising from mathematical models of physical and applied problems. In computational mathematics, however, the mere convergence of an iterative method is not sufficient; it is also important to determine how rapidly the generated sequence approaches the desired solution. This concept, commonly referred to as the rate of convergence, provides an important basis for comparing the efficiency and numerical performance of different iterative schemes.

Over the years, numerous studies have investigated iterative methods for approximating fixed points of self-mappings and non-self-mappings satisfying various contractive conditions in metric, Hilbert, and Banach spaces. While the classical Picard iteration is highly effective for contractions, it may fail to converge when weaker contractive conditions are considered.

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This limitation has motivated the development and investigation of several alternative iterative processes. Among these are the Ishikawa and S-iteration schemes, which are two-step processes, as well as the Noor, SP, and CR iterations, which involve three steps.

Fixed point theory provides a useful framework for addressing a wide range of problems arising in pure and applied mathematics. Since iterative methods offer practical procedures for approximating fixed points, considerable attention has been devoted to studying their convergence, stability, and rate of convergence. In this work, we focus particularly on two three-step iterative processes, namely the Noor iteration and a Modified Noor iteration.

Let X be a complete metric space, let $T: X \rightarrow X$ be a self-mapping, and let

$$F(T) = \{x \in X: Tx = x\} \quad (a)$$

denote the set of fixed points of T . Several iterative processes have been introduced over the years for approximating elements of $F(T)$. The classical Picard iteration is defined by

$$x_{n+1} = Tx_n, \quad n \geq 0 \quad (b)$$

The most fundamental contractive condition in fixed point theory is Banach's contraction condition. An operator satisfying this condition is commonly referred to as a Banach contraction, and the corresponding Picard iteration provides a standard method for approximating its unique fixed point.

Mann (1953) introduced the Mann iterative process, which is given by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 0 \quad (c)$$

where $\{\alpha_n\}$ is a sequence of positive real numbers satisfying the appropriate conditions. The Mann iteration has become one of the most widely studied iterative procedures in fixed point theory.

In 1974, Ishikawa introduced a two-step iterative scheme of the form

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTx_n, \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0 \end{aligned} \quad (d)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying suitable conditions. The Ishikawa iteration can be viewed as a two-step extension of the Mann process and has been extensively studied for various classes of nonlinear mappings. Xue and Zhang (2013) obtained the convergence of implicit Mann and Ishikawa iteration for certain weak generalized contraction mapping in real Banach spaces.

Agarwal et al. (2007) introduced the S-iteration scheme, which is structurally different from both the Mann and Ishikawa processes and is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0 \end{aligned} \quad (e)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the prescribed assumptions. The S-iteration has subsequently received considerable attention because of its convergence properties and, in some settings, its favourable rate of convergence.

Thianwan (2009) introduced another iterative process defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0 \end{aligned} \quad (f)$$

where the associated control sequences satisfy the required conditions. These developments demonstrate the continuing interest in constructing and analysing iterative procedures with improved convergence behaviour. Zamfirescu (1972) obtained fixed point theorem in metric spaces.

Wahab and Rauf (2016) investigated the convergence and rate of convergence of several two-step iterative schemes with errors for Zamfirescu operators in Banach spaces. Their results included a comparison of different iterative processes and indicated the favourable convergence behaviour of the S-iteration under the conditions considered. Earlier, Rhoades and Soltuz (2003) studied the equivalence of Mann and Ishikawa iterations for different classes of operators. They also established equivalence results between modified versions of these iterative schemes under suitable assumptions and further examined the relationship between Mann, Ishikawa, and multistep iterative processes. Equations (a) – (f) are well known one-step and two-step iterative schemes.

The study of three-step iterative procedures has also attracted significant attention. Noor (2000) introduced a three-step iterative scheme and applied it to the approximation of solutions of variational inclusion problems in Hilbert spaces using an updating technique and an auxiliary principle. The Noor iterative scheme is defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)x_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n, \quad n \geq 0 \end{aligned} \quad (1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of positive real numbers satisfying appropriate conditions.

Subsequently, Phuengrattana and Suantai (2011) introduced the SP iteration scheme,

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)z_n + \beta_nTx_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n, \quad n \geq 0 \end{aligned} \quad (2)$$

where the corresponding control sequences satisfy the prescribed assumptions. Renu et al. (2012) further studied the CR iterative scheme, given by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n \\ y_n &= (1 - \beta_n)Tx_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n, \quad n \geq 0 \end{aligned} \quad (3)$$

where the associated parameters are chosen from suitable subsets of the real numbers.

The development of three-step iterative schemes has also been considered in the context of asymptotically nonexpansive mappings. Xu and Noor (2002) introduced and studied a three-step iterative process for approximating fixed points of asymptotically nonexpansive mappings in Banach spaces. Subsequently, Cho et al. (2004) extended this approach to three-step iterative schemes with errors and established weak and strong convergence results for asymptotically nonexpansive mappings in Banach spaces. Schu (2009) also investigated weak and strong convergence properties associated with three-step iterative processes for asymptotically nonexpansive mappings.

Zamfirescu-type operators provide another important class of mappings for which iterative approximation has been extensively investigated. Berinde (2004, 2007) studied the convergence properties of Mann and Ishikawa iterative schemes for Zamfirescu operators in Banach spaces. These results, together with the developments in three-step iterative procedures, provide motivation for further investigation of Noor-type methods under broader contractive conditions.

Motivated by these developments, the present study focuses on the convergence behaviour of the Noor and Modified Noor iterative schemes for Zamfirescu operators in Banach spaces. In particular, we investigate the approximation of fixed points through these three-step procedures and examine their convergence properties under suitable assumptions. The study is intended to extend existing results on Noor-type iterative schemes and to provide further insight into the numerical behaviour of three-step iterative methods for Zamfirescu operators.

Related developments in fixed point theory have also considered more general underlying spaces and contractive structures. For example, Kumar et al. (2015) investigated generalizations of fixed point results in 2-metric spaces, while Duraj and Liftaj (2022) studied common fixed-point results in S -metric spaces. These developments further illustrate the broad scope of fixed point theory and the continuing interest in extending classical fixed point and iterative approximation results to more general mathematical settings

II. SOME USEFUL DEFINITIONS AND CONTRACTION MAPPINGS

The following concepts are important in the proof of our main results.

Theorem 1. (Zamfirescu, 1972). Let (X, d) be a complete metric space and $T: X \rightarrow X$ a mapping for which there exist real numbers a, b and c satisfying $a \in (0, 1), b > 0, c < \frac{1}{2}$, such that for each pair x, y , at least one of the following conditions hold.

$$\begin{aligned} (z_1) \quad & d(T(x), T(y)) \leq ad(x, y) \\ (z_2) \quad & d(T(x), T(y)) \leq b[d(x, T(x)) + d(y, T(y))] \\ (z_3) \quad & d(T(x), T(y)) \leq c[d(x, T(y)) + d(y, T(x))] \end{aligned} \quad (i)$$

Then T has a fixed point p and the Picard iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n, n = 0, 1, 2, \dots$ converge to p for any $x_0 \in X$. System (i) is the Zamfirescu inequalities.

Theorem 2. (Banach Fixed Point Theorem). Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an a -contraction, that is an operator satisfying:

$$d(Tx, Ty) \leq ad(x, y), \text{ for any } x, y \in X \quad (4)$$

with $a \in [0, 1)$ fixed. Then

1. T has a unique fixed point, that is, $F_T = \{x^*\}$;
2. The Picard iteration associated with T , i.e., the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = T(x_{n-1}) = T^n(x_0), \quad n = 1, 2, \dots \quad (5)$$

converges to x^* , for any initial guess $x_0 \in X$;

3. The following apriori and aposteriori error estimates hold:

$$\begin{aligned} d(x_n, x^*) &\leq \frac{a^n}{1-a} \cdot d(x_0, x_1), \quad n = 0, 1, 2, \dots \\ d(x_n, x^*) &\leq \frac{a}{1-a} \cdot d(x_{n-1}, x_n), \quad n = 0, 1, 2, \dots \end{aligned} \quad (6)$$

4. The rate of convergence is given by

$$d(x_n, x^*) \leq a \cdot d(x_{n-1}, x^*) \leq a^n \cdot d(x_0, x^*), \quad n = 1, 2, \dots \quad (7)$$

Proof. There is at most one fixed point, i.e., $\text{card } F_T \leq 1$. Indeed, assuming $x^*, y^* \in F_T, x^* \neq y^*$, since $0 \leq a < 1$, we get the contradiction

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq a \cdot d(x^*, y^*) < d(x^*, y^*)$$

To prove the existence of the fixed point, we will show that, for any given $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Notice that, by (1), we have

$$d(x_2, x_1) = d(Tx_1, Tx_0) \leq ad(x_1, x_0)$$

and by induction,

$$d(x_{n+1}, x_n) \leq a^n d(x_1, x_0), \quad n = 0, 1, 2, \dots \quad (8)$$

Thus, for any numbers $n, p \in \mathbb{N}, p > 0$, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq \sum_{k=n}^{n+p-1} d(x_{k+1}, x_k) \\ &\leq \sum_{k=n}^{n+p-1} a^k d(x_1, x_0) \\ &\leq \frac{a^n}{1-a} d(x_1, x_0) \end{aligned} \quad (9)$$

Since $0 \leq a < 1$, it results that $a^n \rightarrow 0$ (as $n \rightarrow \infty$), which together with (9) shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. But (X, d) is a complete metric space, therefore $\{x_n\}_{n=0}^{\infty}$ converges

to some $x^* \in X$. On the other hand, any Lipschitzian mapping is continuous. Denoting

$$\lim_{n \rightarrow \infty} x_n = x^*$$

we find $x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} T(x_n) = T\left(\lim_{n \rightarrow \infty} x_n\right) = x^*$

which gives $x^* = Tx^*$, i.e., x^* is a fixed point of T . This shows that for any $x_0 \in X$, the Picard iteration converges in X and its limit is a fixed point of T . Since T has at most one fixed point, we deduce that, for every choice of $x_0 \in X$, the Picard iteration converges to the same value x^* , that is, the unique fixed point of T . So we proved (i) and (ii).

To prove (iii) we use (9)

$$d(x_{n+p}, x_n) \leq \frac{a^n}{1-a} d(x_0, x_1) \quad \text{for all } p \in \mathbb{N}^*,$$

and the continuity of the metric and so, by letting $p \rightarrow \infty$, we find

$$\begin{aligned} d(x_n, x^*) &= d(x^*, x_n) = \lim_{n \rightarrow \infty} d(x_{n+p}, x_n) \\ &\leq \frac{a^3}{1-a} \cdot d(x_0, x_1), \quad n \geq 0 \end{aligned}$$

and number (iii) is proved.

To obtain the a posteriori estimation (6), let us notice that by (4), we have

$$d(x_{n+1}, x_n) \leq ad(x_n, x_{n-1})$$

and by induction

$$d(x_{n+k+1}, x_{n-k-1}) \leq a_k d(x_{n-1}, x_{n-1}), \quad k \in \mathbb{N}$$

so

$$\begin{aligned} d(x_{n+p}, x_n) &\leq (a + a^2 + \dots + a^p) d(x_n, x_{n-1}) \\ &\leq \frac{a}{1-a} d(x_n, x_{n-1}) \end{aligned}$$

By letting $p \rightarrow \infty$ the last inequality we get exactly (6). $\square$

Other fixed point theorems of interest are: Brouwer fixed point theorem, Caristi-Kirk fixed point theorem. Also, the following contractive type mappings and their theorems are well known in the literature: Banach Contraction Mapping; Rakotch Contraction Mapping; Edelstein Contraction Mapping; Kannan Contraction Mapping; Bianchini Contraction Mapping; Reich Contraction Mapping; Sehgal Contraction Mapping; Hardy and Rogers Contraction Mapping; Chatterjea Contraction Mapping; Zamfrescu Contraction Mapping; and Imoru and Olatinwo Contraction Mapping. The aim of this work is to ascertain the convergence and convergence rate of some three-step iterative schemes using the Zamfrescu operator in Banach spaces. The objectives are to: test for the convergence of both Noor and modified Noor iterations; and to find out which one of the two iterative schemes converges faster.

Mathematical Formulation

Consider the Picard iteration process defined by the sequence

$$x_{n+1} = Tx_n, \quad n \geq 0 \quad (10)$$

Banach proved the convergence of (10) with the aid of the following contractive mapping:

$$d(Tx, Ty) \leq ad(x, y) \quad (11)$$

where $T: X \rightarrow X$, (X, d) is a metric space, $a \in (0, 1)$, and $x, y \in X$.

The most generalized operator used to approximate fixed points is by Zamfrescu. It was obtained from the Banach, Kannan and Chatterjea contractive mappings as follows. The operator T is called a Kannan mapping if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X \quad (12)$$

Also, the operator T is called a Chatterjea mapping if there exists $c \in (0, \frac{1}{2})$ such that,

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \quad \forall x, y \in X \quad (13)$$

By combining (11), (12) and (13), we have the Zamfrescu operator given, for $x, y \in X$, by

$$\begin{aligned} (z_1) \quad &d(T(x), T(y)) \leq ad(x, y) \\ (z_2) \quad &d(T(x), T(y)) \leq b[d(x, T(x)) + d(y, T(y))] \\ (z_3) \quad &d(T(x), T(y)) \leq c[d(x, T(y)) + d(y, T(x))] \end{aligned} \quad (14)$$

$$\begin{aligned} &d(Tx, Ty) \\ &\leq h\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \end{aligned} \quad (15)$$

where $h = \max\{a, b, c\}$ and $x, y \in X$.

From Theorem 2, it is known that T has a unique fixed point in X , say p . Consider $x, y \in X$. Since T is a Zamfrescu operator, therefore, at least one of the conditions (z_1) , (z_2) and (z_3) is satisfied by T . If (z_2) holds, then

$$\begin{aligned} \|Tx - Ty\| &\leq b[\|x - Tx\| + \|y - Ty\|] \\ &\leq b[\|x - Tx\| + \|y - x\| + \|x - Tx\| \\ &\quad + \|Tx - Ty\|] \\ \Rightarrow (1-b)\|Tx - Ty\| &\leq b\|x - y\| + 2b\|x - Tx\| \\ \Rightarrow \|Tx - Ty\| &\leq \frac{b}{(1-b)}\|x - y\| + 2\frac{b}{(1-b)}\|x - Tx\| \end{aligned} \quad (16)$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \leq \frac{c}{(1-c)}\|x - y\| + 2\frac{c}{(1-c)}\|x - Tx\| \quad (17)$$

Let us denote

$$\lambda = \max\left\{a, \frac{b}{(1-b)}, \frac{c}{(1-c)}\right\} \quad (18)$$

Then we have, $0 \leq \lambda < 1$ and given (z_1) , (16) and (17) yield the following inequality

$$\|Tx - Ty\| \leq \lambda \|x - y\| + 2\lambda \|x - Tx\| \quad (19)$$

holds.

III. MAIN RESULTS

In this section, we state and prove our main results with the aid of Zamfirescu operator and justified with numerical examples.

Theorem 3. Let X be an arbitrary Banach space, B be a non-empty closed convex subset of X , and $T: B \rightarrow B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T , where $F(T)$ denotes the set of fixed points of T . Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by (1) and $x_0 \in B$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the Noor iterative scheme strongly converges to the fixed point $p \in F(T)$.

Proof. Theorem 3 shows that T has a unique fixed point in B , say p . Now let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by (1) and $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $x_0 \in B$ arbitrary. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)x_n + \alpha_n Ty_n - (1 - \alpha_n + \alpha_n)p\| \\ &= \|(1 - \alpha_n)(x_n - p) + \alpha_n(Ty_n - p)\| \quad (20) \\ &\leq \|(1 - \alpha_n)\| \|x_n - p\| + \alpha_n \|Ty_n - p\| \end{aligned}$$

with $x = p$ and $y = y_n$ from (19), we obtain

$$\|Ty_n - p\| \leq \lambda \|y_n - p\| \quad (21)$$

where λ is given by (18). Furthermore, we have

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Tz_n - (1 - \beta_n + \beta_n)p\| \\ &= \|(1 - \beta_n)(x_n - p) + \beta_n(Tz_n - p)\| \quad (22) \\ &\leq \|(1 - \beta_n)\| \|x_n - p\| + \beta_n \|Tz_n - p\| \end{aligned}$$

Again by (19), this time with $x = p$ and $y = z_n$, we find that

$$\|Tz_n - p\| \leq \lambda \|z_n - p\| \quad (23)$$

Combining (21), (22) and (23), we obtain,

$$\|Ty_n - p\| \leq \lambda(1 - \beta_n) \|x_n - p\| + \lambda\beta_n \|z_n - p\| \quad (24)$$

Now,

$$\begin{aligned} \|z_n - p\| &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - (1 - \gamma_n + \gamma_n)p\| \\ &= \|(1 - \gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \quad (25) \\ &\leq \|(1 - \gamma_n)\| \|x_n - p\| + \gamma_n \|Tx_n - p\| \end{aligned}$$

From (24) and (25) we get,

$$\begin{aligned} \|Ty_n - p\| &\leq \lambda[(1 - \beta_n) \|x_n - p\| + \lambda\beta_n[(1 - \gamma_n) \\ &\quad \|x_n - p\| + \gamma_n \|Tx_n - p\|]] \quad (26) \end{aligned}$$

Again by (1), this time with $x = p$ and $y = x_n$ we find that

$$\|Ty_n - p\| \leq \lambda \|x_n - p\| \quad (27)$$

Now, combining (20), (26) and (27) we obtain,

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n[\lambda[(1 - \beta_n) \|x_n - p\| \\ &\quad + \lambda\beta_n[(1 - \gamma_n) \|x_n - p\| + \lambda\gamma_n \|x_n - p\|]]] \\ &= [1 - (1 - \lambda)(1 + \lambda\beta_n + \lambda^2\beta_n\gamma_n)\alpha_n] \|x_n - p\| \\ &\leq [1 - (1 - \lambda)\alpha_n] \|x_n - p\| \quad (28) \end{aligned}$$

From (28),

$$\|x_{n+1} - p\| \leq [1 - (1 - \lambda)\alpha_n] \|x_n - p\|, \quad (29)$$

for $n = 0, 1, 2, \dots$

However, by Mann iterative scheme we inductively obtained

$$\|x_{n+1} - p\| \leq \prod_{k=0}^n [1 - (1 - \lambda)\alpha_k] \|x_0 - p\|, \quad (30)$$

with $n = 0, 1, 2, \dots$

Using the fact that $0 \leq \lambda < 1$, $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we obtain that,

$$\lim_{n \rightarrow \infty} \prod_{k=0}^n [1 - (1 - \lambda)\alpha_k] = 0 \quad (31)$$

Also, from (30) and (31), we obtain $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = 0$. It implies that $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point p . This completes our proof. $\square$

Theorem 4 (modified Noor Iteration). Let X be an arbitrary Banach space, B be a non-empty closed convex subset of X , and $T: B \rightarrow B$ be an operator satisfying the condition Z i.e., $T: B \rightarrow B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T , where $F(T)$ denotes the set of fixed points of T . Let $\{Tx_n\}_{n=0}^{\infty}$ be the Iteration defined by (1) and $x_0 \in B$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the iterative scheme strongly converges to the fixed point $p \in F(T)$.

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n Ty_n \\ y_n &= (1 - \beta_n)x_n + \beta_n Tz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n \quad n \geq 0 \end{aligned} \quad (32)$$

Proof. Theorem 4 shows that T has a unique fixed point in B , say p . Now let $\{Tx_n\}_{n=0}^{\infty}$ be the modified Noor iteration defined by (1) and $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $x_0 \in B$ arbitrary. Then

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)Tx_n + \alpha_n Ty_n - (1 - \alpha_n + \alpha_n)p\| \\ &= \|(1 - \alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \quad (33) \\ &\leq \|(1 - \alpha_n)\| \|Tx_n - p\| + \alpha_n \|Ty_n - p\| \end{aligned}$$

With $x = p$ and $y = y_n$ from (32), we obtain

$$\|Ty_n - p\| \leq \lambda \|y_n - p\| \quad (34)$$

where λ is given by (18). Further, we have

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)Tx_n + \beta_n Tz_n - (1 - \beta_n + \beta_n)p\| \\ &= \|(1 - \beta_n)(Tx_n - p) + \beta_n(Tz_n - p)\| \quad (35) \\ &\leq \|(1 - \beta_n)\| \|Tx_n - p\| + \beta_n \|Tz_n - p\| \end{aligned}$$

By (32), with $x = p$ and $y = z_n$, we find that

$$\|Tz_n - p\| \leq \lambda \|z_n - p\| \quad (36)$$

Combining (34), (35) and (36) we obtain,

$$\|Ty_n - p\| \leq \lambda(1 - \beta_n) \|Tx_n - p\| + \lambda\beta_n \|z_n - p\| \quad (37)$$

Furthermore,

$$\begin{aligned} \|z_n - p\| &= \|(1 - \gamma_n)Tx_n + \gamma_nTx_n - (1 - \gamma_n + \gamma_n)p\| \\ &= \|(1 - \gamma_n)(Tx_n - p) + \gamma_n(Tx_n - p)\| \quad (38) \\ &\leq (1 - \gamma_n) \|Tx_n - p\| + \gamma_n \|Tx_n - p\| \end{aligned}$$

From (37) and (38) we get,

$$\begin{aligned} \|Ty_n - p\| &\leq \lambda[(1 - \beta_n) \|Tx_n - p\| \\ &\quad + \lambda\beta_n[(1 - \gamma_n) \|x_n - p\| + \gamma_n \\ &\quad \|Tx_n - p\|]] \quad (39) \end{aligned}$$

Again by (32), this time with $x = p$ and $y = x_n$ we find that

$$\|Ty_n - p\| \leq \lambda \|Tx_n - p\| \quad (40)$$

Now, combining (33), (39) and (40) we obtain

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 - \alpha_n) \|Tx_n - p\| + \alpha_n[\lambda[(1 - \beta_n) \|Tx_n - p\| \\ &\quad + \lambda\beta_n[(1 - \gamma_n) \|Tx_n - p\| + \lambda\gamma_n \|Tx_n - p\|]]] \\ &= [1 - (1 - \lambda)(1 + \lambda\beta_n + \lambda^2\beta_n\gamma_n)\alpha_n] \|Tx_n - p\| \\ &\leq [1 - (1 - \lambda)\alpha_n] \|Tx_n - p\| \quad (41) \end{aligned}$$

Therefore, from (41),

$$\|x_{n+1} - p\| \leq [1 - (1 - \lambda)\alpha_n] \|Tx_n - p\|, \quad (42)$$

whenever $n = 0, 1, 2, \dots$

Intuitively by Mann iterative scheme we inductively obtain

$$\|x_{n+1} - p\| \leq \prod_{k=0}^n [1 - (1 - \lambda)\alpha_k] \|x_0 - p\|, \quad (43)$$

when $n = 0, 1, 2, \dots$

Using the fact that $0 \leq \lambda < 1$, $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we obtain that,

$$\lim_{n \rightarrow \infty} \prod_{k=0}^n [1 - (1 - \lambda)\alpha_k] = 0 \quad (44)$$

Now, from (43) and (44), we obtain $\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = 0$

Hence, $\{Tx_n\}_{n=0}^{\infty}$ converges strongly to the fixed point p . This completes our proof. $\square$ Here $e_n = \|x_n - q\|$ and the error ratio is $\frac{e_n}{e_{n-1}} = \frac{\|x_n - q\|}{\|x_{n-1} - q\|}$.

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A. Numerical Examples

The following two numerical examples illustrate the convergence behaviour of the two considered iterative schemes and support our analytical results.

Example 1. Let $X = [0, 4]$ and define $T: X \rightarrow X$ by

$$T(x) = (x + 2)^{1/3}.$$

The unique fixed point of T is $q \approx 1.521379707$.

Taking $x_0 = 1.99$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{4}$, the Modified Noor iteration

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_nTy_n, \\ y_n &= (1 - \beta_n)x_n + \beta_nTz_n, \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n, \end{aligned}$$

generates the sequence shown in Table 1. The numerical results demonstrate that $x_n \rightarrow q$, while the error ratio approaches approximately 0.1360316, confirming the linear convergence of the Modified Noor iteration.

Example 2. Let $f: [0, 4] \rightarrow [0, 4]$ be defined by $f(x) = (2x + 3)^{\frac{1}{2}}$ with fixed point $q = 3.0000000$ where the initial guess $x_0 = 4$ and $\alpha_n = \frac{1}{(n+2)}$, $\beta_n = \frac{1}{(n+1)}$, $\gamma_n = \frac{1}{(n+3)}$.

Table 1. Convergence of the Modified Noor Iteration for Example 1

n	x_n	$ x_n - q $	e_n/e_{n-1}
0	1.990000000000	$4.686202932 \times 10^{-1}$	—
1	1.582589676892	$6.120997009 \times 10^{-2}$	0.130617412
2	1.529660283512	$8.280576707 \times 10^{-3}$	0.135281502
3	1.522505279889	$1.125573084 \times 10^{-3}$	0.135929311
4	1.521532804648	$1.530978432 \times 10^{-4}$	0.136017683
5	1.521400532660	$2.082585511 \times 10^{-5}$	0.136029709
6	1.521382539774	$2.832969088 \times 10^{-6}$	0.136031345
7	1.521380092178	$3.853732269 \times 10^{-7}$	0.136031568
8	1.521379759228	$5.242293577 \times 10^{-8}$	0.136031598
9	1.521379713936	$7.131175872 \times 10^{-9}$	0.136031601
10	1.521379707775	$9.700651393 \times 10^{-10}$	0.136031583

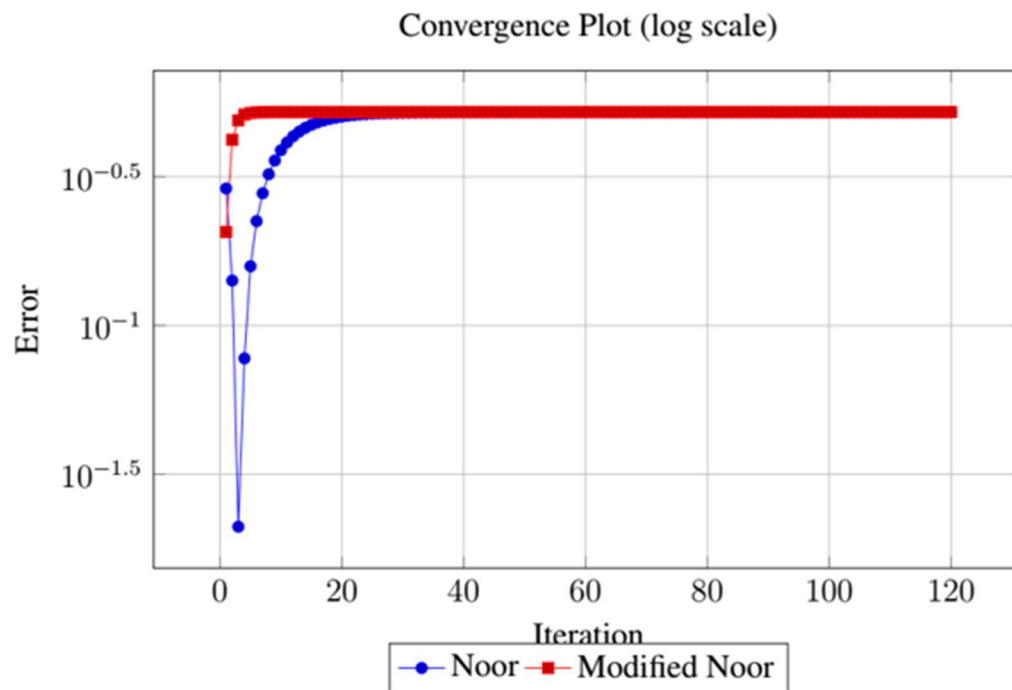


Figure 1: Convergence of Noor and Modified Noor Schemes (log scale error for Example 1

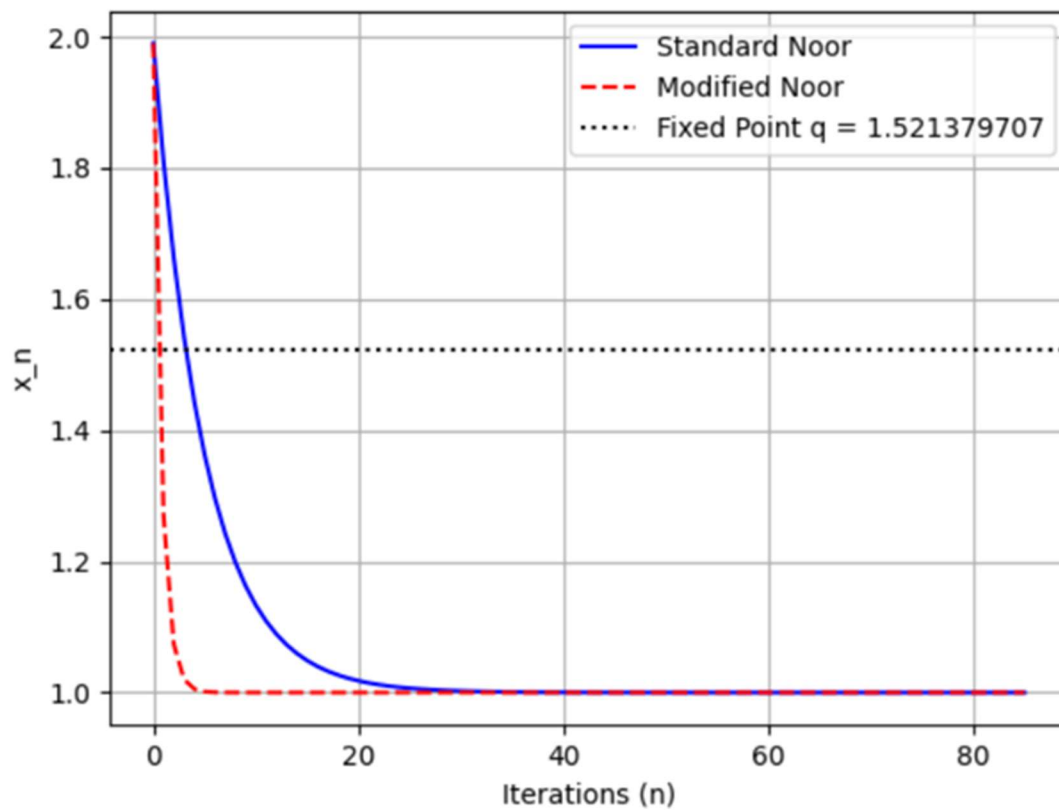


Figure 2: Comparison of Noor and Modified Noor iterations for Example 1

Remark: From example 1, observation on the convergence that the error ratio rapidly approaches $\frac{e_n}{e_{n-1}} \approx 0.1360316$. Thus, the numerical results indicate linear convergence of the Modified Noor iteration for this example. In particular,

$$\lim_{n \rightarrow \infty} \frac{|x_n - q|}{|x_{n-1} - q|} \approx 0.1360316 < 1.$$

This provides a particularly clear numerical demonstration of convergence: the error decreases from approximately 4.69×10^{-1} at $n = 0$ to approximately 9.70×10^{-10} at $n = 10$.

Table 2: Convergence of Noor and Modified Noor Iterative Schemes for Example 2

n	Noor x_n	Noor Error	Modified x_n	Modified Error
1	3.6899457433	0.6899457433	3.1989643246	0.1989643246
2	3.5525915746	0.5525915746	3.0577311284	0.0577311284
3	3.4693322614	0.4693322614	3.0180470209	0.0180470209
4	3.4120655531	0.4120655531	3.0057982930	0.0057982930
5	3.3697169844	0.3697169844	3.0018871675	0.0018871675
6	3.3368614670	0.3368614670	3.0006185904	0.0006185904
7	3.3104785399	0.3104785399	3.0002036442	0.0002036442
8	3.2887348689	0.2887348689	3.0000672312	0.0000672312
9	3.2704453700	0.2704453700	3.0000222393	0.0000222393
10	3.2548061274	0.2548061274	3.0000073669	0.0000073669
11	3.2412504990	0.2412504990	3.0000024429	0.0000024429
12	3.2293664154	0.2293664154	3.0000008107	0.0000008107
13	3.2188463239	0.2188463239	3.0000002692	0.0000002692
14	3.2094555747	0.2094555747	3.0000000895	0.0000000895
15	3.2010117307	0.2010117307	3.0000000297	0.0000000297
16	3.1933706077	0.1933706077	3.0000000099	0.0000000099
17	3.1864166079	0.1864166079	3.0000000033	0.0000000033
18	3.1800558691	0.1800558691	3.0000000011	0.0000000011
19	3.1742113125	0.1742113125	3.0000000004	0.0000000004
20	3.1688189985	0.1688189985	3.0000000001	0.0000000001
21	3.1638254001	0.1638254001	3.0000000000	0.0000000000
22	3.1591853358	0.1591853358	3.0000000000	0.0000000000
23	3.1548603799	0.1548603799	3.0000000000	0.0000000000
24	3.1508176267	0.1508176267	3.0000000000	0.0000000000
:	:	:	:	:
98	3.0610768609	0.0610768609	3.0000000000	0.0000000000
99	3.0606716789	0.0606716789	3.0000000000	0.0000000000
100	3.0602731485	0.0602731485	3.0000000000	0.0000000000

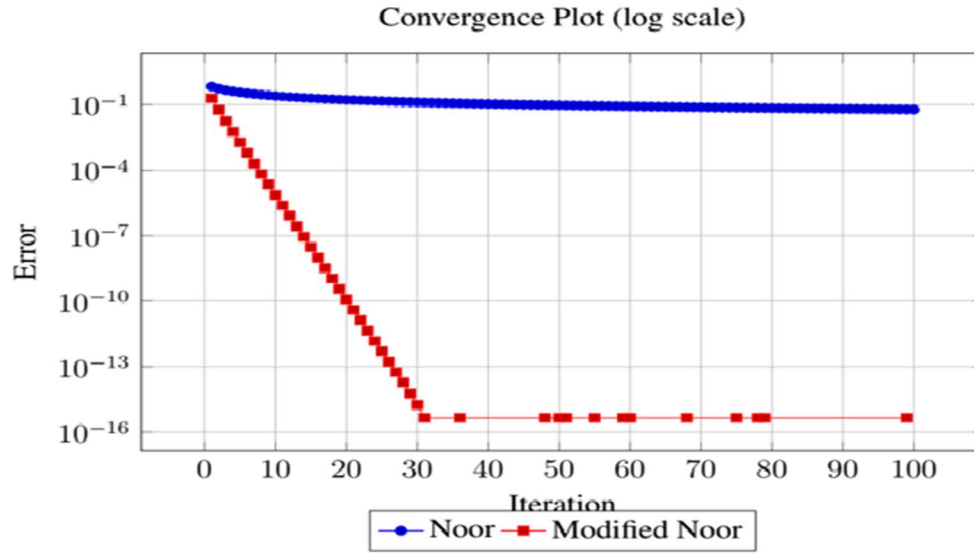


Figure 3: Convergence of Noor and Modified Noor Schemes (log scale error for Example 2

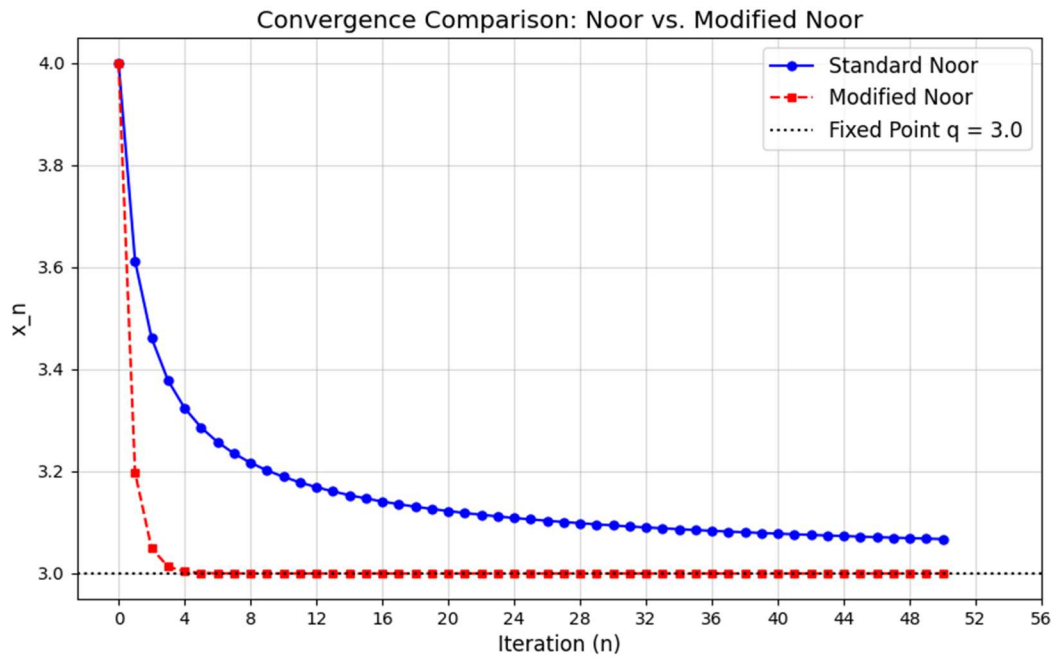


Figure 4: Comparison of Noor and Modified Noor iterations for Example 2

B. Application to a Nonlinear Steady-State Heat-Transfer Model

To illustrate the applicability of the proposed modified Noor iteration to a physical system, we consider a normalized nonlinear steady-state heat-transfer model. Let (x) denote the dimensionless temperature of a lumped thermal system relative to a prescribed reference temperature. Suppose that, at thermal equilibrium, the nonlinear heat-loss mechanism is balanced by a linear heat-transfer contribution and a constant heat input. After suitable normalization of the physical

parameters, the steady-state energy balance can be represented by

$$x^3 = x + 2$$

Consequently, the steady-state problem can be written in the fixed-point form

$$x = T(x) \text{ where } T(x) = (x + 2)^{1/3}.$$

We consider the closed interval $X = [0, 4]$. Clearly, $T \rightarrow X$ and the steady-state temperature corresponds to the fixed point q satisfying

$$q = T(q) = (q + 2)^{\frac{1}{3}}$$

Equivalently,

$$q^3 - q - 2 = 0$$

whose positive solution is

$$q = 1.5213797068.$$

Thus, q represents the normalized steady-state temperature of the system. If the dimensional temperature is defined by $\theta = \theta_a + \theta_{sx}$, then q corresponds to the physical temperature $\theta^* = \theta_a + \theta_{sq}$. For the numerical experiment, we take

$$x_0 = 1.99, \alpha_n = \beta_n = \gamma_n = \frac{1}{4}$$

and apply both the original Noor iteration and the proposed modified Noor iteration to the same fixed-point problem. The resulting approximations and their absolute errors with respect to the steady-state value q are presented in Tables 1 and 2.

The numerical results show a clear difference in the convergence behavior of the two schemes. Starting from the same initial approximation $x_0 = 1.99$, the modified Noor iteration produces $x_3 = 1.522505280$ with an absolute error of approximately 1.13×10^{-3} whereas the original Noor iteration gives $x_3 = 1.741430025$ with an error of approximately 2.20×10^{-1} . At the ninth iteration, the error of the modified Noor iteration decreases to approximately 7.13×10^{-9} while the corresponding error for the original Noor iteration remains approximately 4.87×10^{-3} .

These computations indicate that, for the nonlinear thermal model considered here, the proposed modified Noor scheme reaches the steady-state solution considerably faster than the original Noor scheme. The example therefore provides a physical interpretation of the fixed-point problem and illustrates the potential usefulness of the proposed modification for numerical approximation of nonlinear steady states in thermal and related engineering systems.

IV. CONCLUSION

This study investigated the convergence and rate of convergence of selected three-step iterative schemes in Banach spaces with particular emphasis on the Noor and Modified Noor iterations. The numerical results presented in Example 1 show that the Modified Noor iteration converges faster than the Noor iteration. Similarly, the results in Example 2 demonstrate that the Modified Noor iteration exhibits a more favourable convergence rate and requires fewer iterations to attain the prescribed level of accuracy. It is important to note that the convergence behaviour of an iterative scheme depends, among other factors, on the choice of the control parameters α_n , β_n , and γ_n . Thus, the observed numerical advantage of the Modified Noor iteration is associated with the specific parameter choices and formulations considered in this study. Overall, the numerical experiments indicate that the Modified Noor iteration provides an effective alternative to the Noor iteration for approximating fixed points under the

conditions considered. These findings suggest that the Modified Noor scheme may be a useful three-step iterative method for fixed-point computations, particularly when convergence speed and computational efficiency are important considerations. Further studies may investigate the behaviour of these schemes under different classes of operators, parameter choices and error conditions.

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AUTHOR CONTRIBUTIONS

M. A. Adeniyi: Formal analysis, Investigation, Writing—original draft, Funding acquisition, Investigation, Resources, editing, Conceptualization and Data curation. **M. O. Fatai:** Methodology, Resources, Resources, editing, Investigation and Data curation. **Richmond U. Kanu:** Formal analysis, Resources, Funding acquisition, Investigation, Resources, Conceptualization and Data curation. **E. E. Aribike:** Formal analysis, Resources, Funding acquisition, Investigation, Resources, Conceptualization and editing. **K. Rauf:** Validation, Software, Visualization and Project administration, Supervision, review and editing.

Competing interests

The manuscript was read and approved by all the authors. They therefore declare that there is no conflicts of interest.

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