

A Comparative Analysis of Simply Supported and Clamped-Free Beam Models with Moving Masses on Pasternak Elastic Foundations



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ABSTRACT: In this study, we develop a unified, high-fidelity framework to predict the transient dynamics of Euler–Bernoulli beams on a Pasternak elastic foundation subjected to moving masses, explicitly retaining the inertia coupling between the travelling mass patch and the host structure. Spatial discretization is performed with a Legendre–Galerkin projection that enforces boundary conditions exactly and delivers spectral convergence for smooth spatial fields; time integration uses an energy-consistent implicit Newmark– β algorithm that handles the resulting time-varying generalized mass matrix and inertial correction terms without loss of stability. The formulation is applied to two practically important boundary scenarios: simply supported and clamped–free (cantilever) to expose how support constraints alter modal participation, peak deflections, and critical-speed behavior. Parametric studies across foundation moduli (Winkler, Pasternak shear), flexural rigidity, cross-section, mass per unit length and moving-mass width/speed reveal three robust findings: moving mass smooths spatial response but shifts natural frequencies and induces strong mode coupling near critical speeds, Pasternak shear coupling markedly reduces local peaks and elevates critical speeds relative to Winkler support; and boundary condition strongly modulates sensitivity. Cantilevers concentrate response near the free end and are more sensitive to local reinforcement, whereas simply supported spans distribute effects more uniformly. The method is validated against limiting analytic cases and converges rapidly with few modes, offering a practical, reproducible toolset for designers and researchers confronting heavier, faster, and spatially extended moving masses.

KEYWORDS: Moving distributed mass, Pasternak foundation, Legendre–Galerkin, Newmark– β , Simply supported, Clamped-free.

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I. INTRODUCTION

Modern infrastructure and mechanical systems increasingly operate under conditions in which substantial masses move rapidly and continuously across structural elements. Examples range from high-speed trains and heavy freight vehicles on bridges to tracked machinery and conveyor-mounted equipment on industrial girders. In such applications the moving body is often spatially extended and carries non-negligible inertia; a combination that produces time-varying mass distributions, inertia coupling, modal energy exchange and speed-dependent amplification that cannot be captured adequately by classical point-load models. Predicting and controlling these phenomena is critical for serviceability, safety and longevity: underestimated dynamic amplification can trigger excessive deflections, fatigue, or uncomfortable vibrations, over-conservative designs waste material and cost.

Capturing the essential physics of moving distributed masses demands three modeling ingredients. First, the continuum formulation must retain the time-dependent mass density and its inertial coupling to the structural acceleration field. Second, a high-fidelity spatial discretization is required to represent smooth distributed movers and to resolve higher-mode content with minimal modal truncation error; spectral and high-order modal bases are attractive because they converge rapidly for smooth fields. Third, time

integration must treat non-autonomous mass operators and the associated inertial corrections consistently to avoid spurious energy growth or dissipation, especially near resonant regimes. Moreover, structural support conditions fundamentally shape response: simply supported spans and cantilevers (clamped–free) exhibit markedly different mode shapes, natural frequencies, and spatial sensitivity to local or distributed changes in stiffness, mass or foundation properties. A transferable predictive framework must therefore accommodate both boundary archetypes to be broadly useful to engineers and researchers.

The dynamics of beams under moving excitations is a foundational topic in structural dynamics, spanning classic theory, computational methods, and contemporary applications such as high-speed rail, heavy vehicle–bridge interaction, and industrial conveyors. Early analytical studies established the importance of boundary conditions and system size on moving-load response and identified critical velocities where classical steady-state approximations fail. Time-integration framework was applied to compute time histories and validated semi-analytic and numerical schemes (Newmark 1959). A series solution for a finite beam with a moving concentrated load which set an early benchmark for finite-span behaviour and showed how support conditions (e.g., simply supported ends) strongly shaped spatial resonance and dynamic amplification was adopted (Steele 1967).

A fundamental modeling distinction emerged between moving forces (prescribed external transverse loads) and moving masses (traveling bodies whose inertia couples to the structure). Moving-mass formulations are intrinsically non-autonomous: the effective mass density of the beam becomes time-dependent, and inertial coupling terms must be retained for accuracy. Ichikawa *et al.* and subsequent authors demonstrated that moving masses can shift natural frequencies, produce convective inertia effects, and alter critical-speed thresholds; phenomena that cannot be captured by simple moving-force idealizations (Ichikawa *et al.*, 2000; Nikkhoo *et al.*, 2007). Experimental and theoretical studies on damaged or multi-span beams also show that moving masses excite higher modes and can substantially amplify local responses relative to equivalent moving loads (Bilello and Bergman, 2004; Kiani *et al.*, 2009).

Foundation modeling significantly modulates moving-load/mass effects. A Winkler foundation (local springs) captures vertical support but neglects shear coupling between adjacent points; Pasternak's two-parameter model (Winkler and shear layer) provides a minimal yet highly effective correction that changes both local stiffness and wave propagation characteristics. Several works have shown that Pasternak shear coupling reduces local deflection peaks and raises effective critical velocities relative to Winkler-only models, particularly important for high-speed movers and for cantilevered systems where curvature concentrates near free ends (Dimitrovová, 2016). A study on the transient behavior of Rayleigh beams on biparametric foundation foundations subjected to moving loads via the Rayleigh-Ritz and Runge-Kutta methods was conducted (Adeoye *et al.*, 2025).

From a numerical standpoint, spatial discretization and time integration must be chosen with care. Spectral and spectral-Galerkin methods (Legendre, Chebyshev, or spectral-element formulations) provide very rapid convergence for smooth spatial kernels such as Gaussian or raised-cosine moving patches and are thus attractive when the mover is spatially distributed rather than point-like (Wu, 2007; Azizi, *et al.*, 2012). Legendre-Galerkin matrix (LGM) approach was applied to solve structural dynamics problems and showed excellent accuracy per degree of freedom, making it well suited to reduced-order, repeated-assembly tasks encountered when updating each time step (Momeni *et al.*, 2021). Spectral-element and high-order modal approaches have equally been shown to capture both transient and steady harmonic content of moving-load problems with reduced dispersion error (Azizi, *et al.*, 2012; Basu, 2013).

Time integration methods must correctly treat non-stationary mass matrices and any consistent inertial correction terms. The implicit Newmark- β method (average acceleration) is widely used for such problems because of unconditional stability in the linear regime and compatibility with non-proportional damping; several authors have described consistent procedures for updating and computing corrected effective loads to avoid spurious energy injection or dissipation (Wu, *et al.*, 2007). An approximate solution to Euler-Bernoulli beams on a two parameter foundation was studied by adopting Legendre polynomial approach for spatial discretization and Newmark-beta technique for the matrix equation's equation solver (Adeoye *et al.*, 2026).

Boundary conditions qualitatively change sensitivity to parameters. Simply supported (SS) spans tend to distribute bending energy and therefore show more uniform sensitivity to foundation stiffness or cross-section changes, while clamped-free (CF) (cantilever) systems localize curvature near the free end and thus respond more strongly to local load intensity, cross-section, or foundation changes near that end. The analysis of the response of Euler-Bernoulli beams with elastically supported ends under the influence of moving distributed masses was discussed (Adeoye and Akintomide, 2017). Several comparative numerical and experimental studies highlight this contrast: designers must therefore select mitigation strategies (local reinforcement, foundation stiffening, damping placement) with boundary condition in mind (Liu and Huang, 1988).

Recent work emphasizes realistic moving-mass modeling (finite-width patches, smooth kernels), high-order spatial discretization for spectral accuracy, and energy-consistent implicit time stepping. Combined parametric studies examining Pasternak shear, Winkler stiffness, flexural rigidity, cross-section, mass, damping, load intensity and speed show consistent trends: increasing stiffness or shear support reduces amplitude and raises critical speeds, increasing damping attenuates resonant peaks, and distributed movers smooth spatial peaks relative to point masses but can still cause significant modal coupling at specific speeds (Pourzeynali *et al.*, 2021). These results motivate a unified framework that uses Legendre-Galerkin spatial projection and Newmark- β time integration to capture both SS and CF boundary behaviors across parameter regimes; a gap that recent studies are beginning to close (Saito and Teresawa, 1980; Basu, 2013).

II. MATHEMATICAL FORMULATIONS

A. Kinematics and Assumptions

For the problem kinematics, we have the following parameters: transverse displacement $w(x, t)$, beam length $x \in [0, l]$, beam mass per unit length m'_b , moving distributed mass per unit length $m_m(x, t)$, total instantaneous mass density $\mu(x, t) = m'_b + m_m(x, t)$, viscous damping per unit length C , flexural rigidity EI , Pasternak foundation: Winkler modulus K_0 and shear parameter G_0 . The problem is formulated as follows:

1) Bending Moment and Shear Force

The bending moment expression is defined as

$$M(x, t) = -EI \frac{\partial^2 w}{\partial x^2} \quad (1)$$

Similarly the shear force is given as

$$V(x, t) = \frac{\partial M}{\partial x} = -EI \frac{\partial^3 w}{\partial x^3} \quad (2)$$

2) Vertical Equilibrium on a Beam Element

Consider an infinitesimal element $[x + dx]$. The equilibrium of forces is expressed as

$$-\frac{\partial V}{\partial x} + q + q_f = \mu(x, t) \frac{\partial^2 w}{\partial t^2} + \frac{\partial \mu}{\partial t} \left(\frac{\partial w}{\partial t} \right) \quad (3)$$

The term involving $\frac{\partial \mu}{\partial t}$ appears when the mass distribution varies with time (moving mass).

3) Pasternak Foundation Reaction

The Pasternak foundation reaction includes the Winkler and shear foundation reaction per unit length as expressed in the equation

$$q_f(x, t) = K_o w - G_o \frac{\partial^2 w}{\partial x^2} \quad (4)$$

Substituting $\frac{\partial v}{\partial x}$ and q_f into the equilibrium equation (3) yields:

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} + K_o w - G_o \frac{\partial^2 w}{\partial x^2} + q \\ = \mu(x, t) \frac{\partial^2 w}{\partial t^2} + \frac{\partial \mu}{\partial t} \left(\frac{\partial w}{\partial t} \right) \end{aligned} \quad (5)$$

On further inclusion of viscous damping term $\frac{\partial w}{\partial t}$, we obtain the full partial differential equation governing the problem as

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} + C \frac{\partial w}{\partial t} + K_o w - G_o \frac{\partial^2 w}{\partial x^2} + q \\ = \mu(x, t) \frac{\partial^2 w}{\partial t^2} + \frac{\partial \mu}{\partial t} \left(\frac{\partial w}{\partial t} \right) \end{aligned} \quad (6)$$

Substituting the expression for $\mu(x, t)$ we have

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} + C \frac{\partial w}{\partial t} + K_o w - G_o \frac{\partial^2 w}{\partial x^2} + q \\ = [m'_b + m_m(x, t)] \frac{\partial^2 w}{\partial t^2} \\ + \frac{\partial}{\partial t} (m'_b \\ + m_m(x, t)) \left(\frac{\partial w}{\partial t} \right) \end{aligned} \quad (7)$$

Assuming $m_m(x, t) = 0$ and $\frac{\partial}{\partial t} m'_b = 0$ because it is independent of time t , we have

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} + m_b \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} + K_o w - G_o \frac{\partial^2 w}{\partial x^2} \\ = -q \end{aligned} \quad (8)$$

B. Problem Formulation

The equation of motion for the deflection $w(x, t)$ of the beam is given by:

$$\begin{aligned} EI \frac{\partial^4 w}{\partial x^4} + m_b \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} + K_o w - G_o \frac{\partial^2 w}{\partial x^2} \\ - u_m(x - vt) \\ \left[1 - \frac{1}{g} \left(\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial x \partial t} + v^2 \frac{\partial^2 w}{\partial x^2} + v \frac{\partial w}{\partial x} \right) \right] = 0 \end{aligned} \quad (9)$$

where EI is flexural rigidity of the beam, m_b is mass per unit length of the beam, C is damping coefficient (one-parameter damping), K_o is Winkler foundation modulus, G_o is Pasternak shear layer modulus, $u_m(x - vt) = P_o \delta(x - vt)$ is the moving mass and v is velocity of the traversing load. For this research $\dot{v}$ is set to zero which means that the model only considered constant speed v and neglected any acceleration of the mass itself. This means that the moving mass is assumed to traverse the beam at a prescribed constant velocity v and makes the inertia operator to be reduced to the usual convective acceleration term. Therefore, additional curvature effects are neglected under the small deflection Euler-Bernoulli assumption.

The moving mass is modelled as a Dirac delta distribution located at the instantaneous position $x = vt$. The expression for the Delta function is expanded as

$$\delta(x - vt) = \frac{1}{l} \left[1 + 2 \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \cos \frac{k\pi x}{l} \right] \quad (10)$$

So the functional form is a point-like moving distribution centred at $x = vt$ with total intensity P_o while the effective width in this model is zero. It is an idealized concentrated mass, not a finite-width patch. The Fourier series form is just the mathematical representation of that delta distribution over the beam length l .

In general, $EI \frac{\partial^4 w}{\partial x^4}$ is bending stiffness of the beam, $m_b \frac{\partial^2 w}{\partial t^2}$ is inertia of the beam, $C \frac{\partial w}{\partial t}$ is one-parameter damping effect (ie. velocity-proportional damping term), $K_o w$ is Winkler foundation restoring force, $G_o \frac{\partial^2 w}{\partial x^2}$ is shear layer interaction of Pasternak foundation and $u_m(x - vt) \left(\frac{\partial^2 w}{\partial t^2} + 2v \frac{\partial^2 w}{\partial x \partial t} + v^2 \frac{\partial^2 w}{\partial x^2} \right)$ is inertia effect due to the moving mass. This equation governs the behaviour of the Euler-Bernoulli beam under the action of a moving mass, one-parameter damping and Pasternak foundation effects.

III. MATERIALS AND METHODS

The governing equation of the Euler-Bernoulli beam with one-parameter damping under moving distributed masses on a Pasternak foundation for this research is solved using Legendre polynomials and newmark-beta method.

A. Legendre Polynomials

We assume the approximate displacement function $w(x, t)$ using legendre polynomials of the form

$$w(x, t) \approx \sum_{n=0}^N a_n(t) P_n(\epsilon) \quad (11)$$

where $P_n(\epsilon)$ is the Legendre polynomial of degree n , $\epsilon = \frac{2x}{l} - 1$, maps x from $[0, l]$ to $[-1, 1]$ and $a_n(t)$ are the unknown time-dependent coefficients.

After the mapping, spatial derivatives with respect to x are converted using $\frac{d}{dx} = \frac{2}{l} \frac{d}{d\epsilon}$, $\frac{\partial^2}{\partial x^2} = \left(\frac{2}{l}\right)^2 \frac{\partial^2}{\partial \epsilon^2}$, $\frac{\partial^4}{\partial x^4} = \left(\frac{2}{l}\right)^4 \frac{\partial^4}{\partial \epsilon^4}$ and the higher derivatives are obtained from the Legendre basis functions.

Substituting (2) and (3) into the governing equation, differentiating, and simplifying, we have

$$\begin{aligned} EI \sum_{j=0}^N a_j(t) \frac{\partial^4 P_j(\epsilon)}{\partial \epsilon^4} \left(\frac{2}{l}\right)^4 + m_b \sum_{n=0}^N \ddot{a}_j(t) P_j(\epsilon) \\ + C \sum_{n=0}^N \dot{a}_j(t) P_j(\epsilon) + \\ K_o \sum_{j=0}^N a_j(t) P_j(\epsilon) - G_o \sum_{j=0}^N a_j(t) \frac{\partial^2 P_j(\epsilon)}{\partial \epsilon^2} \left(\frac{2}{l}\right)^2 \\ + \frac{P_o}{gl} \left[1 + 2 \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \cos \frac{k\pi x}{l} \right] \end{aligned}$$

$$\begin{aligned}
& \left(2v \sum_{j=0}^N \dot{a}_j(t) \frac{\partial P_j(\epsilon)}{\partial \epsilon} \left(\frac{2}{l} \right) \right. \\
& \quad \left. + v^2 \sum_{j=0}^N a_j(t) \frac{\partial^2 P_j(\epsilon)}{\partial \epsilon^2} \left(\frac{2}{l} \right)^2 \right) \\
& = P_o \sum_{j=0}^N P_j(\epsilon) \quad (13)
\end{aligned}$$

To obtain the finite system, the residual is multiplied by a test function $P_i(\epsilon)$ and integrated using $x = \frac{l}{2}(\epsilon + 1)$ over ζ from -1 to 1 , we have

$$\begin{aligned}
& \sum_{j=0}^N \left[m_b \delta_{ij} \ddot{a}_j(t) + C \delta_{ij} \dot{a}_j(t) \right. \\
& \quad \left. + \left(EI \left(\frac{2}{l} \right)^4 I_{ij}^{(4)} + K_o \delta_{ij} - G_o \left(\frac{2}{l} \right)^2 I_{ij}^{(2)} \right) a_j(t) \right] \\
& + \frac{P_o}{gl} \left(2v \sum_{j=0}^N \left(\frac{2}{l} \right) I_{ij}^{(1)} \dot{a}_j(t) + v^2 \sum_{j=0}^N \left(\frac{2}{l} \right)^2 I_{ij}^{(2)} a_j(t) \right) + \frac{2P_o}{gl} \\
& \quad \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \left(2v \sum_{j=0}^N \left(\frac{2}{l} \right) E_{ij}^{(1)} \dot{a}_j(t) \right. \\
& \quad \left. + v^2 \sum_{j=0}^N E_{ij}^{(2)} \left(\frac{2}{l} \right)^2 a_j(t) \right) \\
& = P_o \sum_{j=0}^N \delta_{ij} \quad (14)
\end{aligned}$$

where

$$\begin{aligned}
I_{ij}^{(4)} &= \int_{-1}^1 P_i(\epsilon) \frac{d^4 P_j(\epsilon)}{d\epsilon^4} d\epsilon, & I_{ij}^{(2)} &= \int_{-1}^1 P_i(\epsilon) \frac{d^2 P_j(\epsilon)}{d\epsilon^2} d\epsilon, \\
I_{ij}^{(1)} &= \int_{-1}^1 P_i(\epsilon) \frac{\partial P_j(\epsilon)}{\partial \epsilon} d\epsilon, & E_{ij}^{(1)} &= \int_{-1}^1 \cos k\pi(\epsilon + 1) P_i(\epsilon) \frac{\partial P_j(\epsilon)}{\partial \epsilon} d\epsilon, \\
& & E_{ij}^{(2)} &= \int_{-1}^1 \cos k\pi(\epsilon + 1) P_i(\epsilon) \frac{d^2 P_j(\epsilon)}{d\epsilon^2} d\epsilon, & F_{ij} &= \int_{-1}^1 P_o P_i(\epsilon) d\epsilon \text{ and } \delta_{ij} \text{ is the} \\
& & & & & \text{Kronecker delta, defined as} \\
& & & & & \delta_{ij} \\
& & & & & = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \quad (15)
\end{aligned}$$

Equation (14) is further simplified as

$$\begin{aligned}
& \sum_{j=0}^N \left[m_b \delta_{ij} \ddot{a}_j(t) \right. \\
& \quad \left. + \left(C \delta_{ij} + \frac{2vP_o}{gl} \sum_{j=0}^N \left(\frac{2}{l} \right) I_{ij}^{(1)} \right. \right. \\
& \quad \left. \left. + \frac{4vP_o}{gl} \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \sum_{j=0}^N \left(\frac{2}{l} \right) E_{ij}^{(1)} \right) \dot{a}_j(t) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^N \left(EI \left(\frac{2}{l} \right)^4 I_{ij}^{(4)} + K_o \delta_{ij} - G_o \left(\frac{2}{l} \right)^2 I_{ij}^{(2)} \right. \\
& \quad \left. + \frac{v^2 P_o}{gl} \sum_{j=0}^N \left(\frac{2}{l} \right)^2 I_{ij}^{(2)} \right) a_j(t) \\
& + \frac{2v^2 P_o}{gl} \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \sum_{j=0}^N E_{ij}^{(2)} \left(\frac{2}{l} \right)^2 a_j(t) \\
& = \sum_{j=0}^N P_o \delta_{ij} \quad (16)
\end{aligned}$$

The resulting system of ODEs after factorization becomes

$$\begin{aligned}
& \mathbf{M} \cdot \ddot{\mathbf{a}}(t) + \mathbf{C}^* \cdot \dot{\mathbf{a}}(t) + \mathbf{K}^* \cdot \mathbf{a}(t) \\
& = \mathbf{F}(t) \quad (17)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{M} &= m_b \delta_{ij}, & \mathbf{C}^* &= C \delta_{ij} + \frac{2vP_o}{gl} \sum_{j=0}^N \left(\frac{2}{l} \right) I_{ij}^{(1)} + \\
& \frac{4vP_o}{gl} \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \sum_{j=0}^N \left(\frac{2}{l} \right) E_{ij}^{(1)}, \\
\mathbf{K}^* &= EI \left(\frac{2}{l} \right)^4 I_{ij}^{(4)} + K_o \delta_{ij} - G_o \left(\frac{2}{l} \right)^2 I_{ij}^{(2)} + \\
& \frac{v^2 P_o}{gl} \sum_{j=0}^N \left(\frac{2}{l} \right)^2 I_{ij}^{(2)} + \frac{2v^2 P_o}{gl} \sum_{k=1}^{\infty} \cos \frac{k\pi vt}{l} \sum_{j=0}^N E_{ij}^{(2)} \left(\frac{2}{l} \right)^2 \\
\mathbf{a}(t) &= [a_0(t), a_1(t), \dots, a_N(t)]^T
\end{aligned}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ defined as:

$\mathbf{M}$ is the mass matrix (including both the beam mass and any contributions from the moving distributed mass), $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix (which also includes contribution from the beam bending and foundation effects), and $\mathbf{F}(t)$ is the forcing vector (accounting for the moving mass inertia effects).

Applying orthogonality conditions, the matrix representations of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given as follows:

- **Mass matrix $\mathbf{M}$:** The beam inertia is given by the term $m_b \frac{\partial^2 w}{\partial t^2}$
- $$M_{ij} = \frac{l}{2} m_b \int_{-1}^1 P_i(\epsilon) P_j(\epsilon) d\epsilon \quad (18)$$

Since the Legendre polynomial satisfy

$$\int_{-1}^1 P_i(\epsilon) P_j(\epsilon) d\epsilon = \frac{2}{2j+1} \delta_{ij} \quad (19)$$

Therefore,

$$M_{ij} = \frac{l}{2j+1} m_b \text{ and } M_{ij} = 0 \quad (i \neq j) \quad (20)$$

- **Damping Matrix $\mathbf{C}$:** A one-parameter (viscous) damping term enters as $C \frac{\partial w}{\partial t}$

$$C_{ij} = \frac{l}{2} C \int_{-1}^1 P_i(\epsilon) P_j(\epsilon) d\epsilon = \frac{l}{2j+1} C \delta_{ij} \quad (21)$$

where C is a damping coefficient associated with the continuous medium, foundation, or surrounding viscous resistance.

1. **Stiffness Matrix $\mathbf{K}$:** The stiffness contributions come from three parts:

- **Beam Bending:** The bending term is $EI \frac{\partial^4 w}{\partial x^4}$

$$\text{Recall if } \frac{d}{dx} = \frac{2}{l} \frac{d}{d\epsilon} \text{ then } \frac{d^4 P_j}{dx^4} = \left(\frac{2}{l} \right)^4 \frac{d^4 P_j}{d\epsilon^4}$$

$$K_{ij}^{(b)} = \frac{l}{2} EI \left(\frac{2}{l} \right)^4 \int_{-1}^1 \frac{d^4 P_j}{d\epsilon^4} P_i(\epsilon) d\epsilon \quad (22)$$

- Pasternak Shear Layer: The Pasternak foundation adds a term $G_0 \frac{\partial^2 w}{\partial x^2}$
 $\frac{d^2 P_j}{dx^2} = \left(\frac{2}{l} \right)^2 \frac{d^2 P_j}{d\epsilon^2}$, $K_{ij}^{(s)} = \frac{l}{2} G_0 \left(\frac{2}{l} \right)^2 \int_{-1}^1 \frac{d^2 P_j}{d\epsilon^2} P_i(\epsilon) d\epsilon$ (23)
- Winkler Foundation: The Winkler part contributes $K_0 w$

$$K_{ij}^{(w)} = \frac{l}{2} K_0 \int_{-1}^1 P_i(\epsilon) P_j(\epsilon) d\epsilon = \frac{l}{2j+1} K_0 \delta_{ij} \quad (24)$$

Hence, the total stiffness matrix gives

$$K_{ij} = K_{ij}^{(b)} - K_{ij}^{(s)} + K_{ij}^{(w)} \quad (25)$$

That is,

$$K_{ij} = \left[\frac{l}{2} EI \left(\frac{2}{l} \right)^4 \int_{-1}^1 \frac{d^4 P_j}{d\epsilon^4} P_i(\epsilon) d\epsilon - G_0 \left(\frac{2}{l} \right)^2 \int_{-1}^1 \frac{d^2 P_j}{d\epsilon^2} P_i(\epsilon) d\epsilon + K_0 \int_{-1}^1 P_i(\epsilon) P_j(\epsilon) d\epsilon \right] \quad (26)$$

1. *Forcing Vector $F(t)$* : The forcing vector comes from the moving mass

$$F_i(t) = \frac{l}{2} \int_{-1}^1 P_0 P_i(\epsilon) d\epsilon \quad (27)$$

Thus, the full forcing vector is assembled as

$$F(t) = [F_0(t), F_1(t), \dots, F_N(t)]^T \quad (28)$$

For $i = 0, 1, \dots, N$ in all the integrals. The second order system of ODEs in equation (17) is solved by the Newmark-beta method.

B. Boundary Conditions

The beam boundary conditions are imposed on the Legendre expansion by enforcing the kinematic and natural end conditions at the mapped endpoints of the domain. Simply supported means the beam can rotate freely at the supports but cannot translate vertically. Moments and transverse deflections are zero at the supports, $x = 0$ and $x = l$. Mathematically, we have

For transverse deflections

$$w(0, t) = w(l, t) = 0 \quad (29)$$

For bending moments

$$\begin{aligned} M(0, t) &= -EI \frac{\partial^2}{\partial x^2} w(0, t) = 0 \\ M(l, t) &= -EI \frac{\partial^2}{\partial x^2} w(l, t) = 0 \end{aligned} \quad (30)$$

Similarly, for clamped free boundary conditions, the clamped (fixed) end cannot translate or rotate while the free end has no moment or shear applied (both zero)

At the clamped end $x = 0$;

$$w(0, t) = 0 \quad (\text{zero deflection})$$

$$\frac{\partial}{\partial x} w(0, t) = 0 \quad (\text{zero slope and rotation prevented}) \quad (31)$$

At the free end $x = l$;

$$M(l, t) = EI \frac{\partial^3}{\partial x^3} w(l, t) = 0 \quad (\text{zero bending moment})$$

$$V(l, t) - EI \frac{\partial^3}{\partial x^3} w(l, t) = 0 \quad (\text{zero shear force}) \quad (32)$$

These constraints are added to the reduced modal system prior to the time integration step. When properly applied, they also improve the Legendre polynomial approximation.

C. Newmark-beta Method

The Newmark-beta method provides a time-stepping scheme for solving these equations. The idea is to approximate the acceleration and velocity at the next time step $t_{j+1} = t_j + \Delta t$ as

$$\begin{aligned} a_{j+1} &= a_j + \Delta t \cdot \dot{a}_j + \Delta t^2 \left[\left(\frac{1}{2} - \beta \right) \ddot{a}_j + \beta \ddot{a}_{j+1} \right] \\ \dot{a}_{j+1} &= \dot{a}_j + \Delta t \left[(1 - \gamma) \ddot{a}_j + \gamma \ddot{a}_{j+1} \right] \end{aligned} \quad (33)$$

where β and γ are Newmark-beta parameters.

Their common choices of values are $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$ (the average acceleration method), which provide unconditional stability for linear problems. This is to be noted that the semi discrete equation obtained does not only contain mass matrix, damping matrix and stiffness matrix so the usual expressions will change while forming the effective stiffness matrix. The application of Newmark-beta method requires the following steps:

(i) Initialization Phase

We set initial conditions for displacement a_0 and velocity $\dot{a}_0$ and then compute the initial acceleration by

$$\ddot{a}_0 = M^{-1} [F_0 - C^* \dot{a}_0 - K^* a_0] \quad (34)$$

(ii) Time-Stepping Loop

For each time step $j = 0, 1, 2, \dots$, we perform the following:

a. Predictor Step

Trial displacement a_{j+1}^{trial} and trial velocity $\dot{a}_{j+1}^{\text{trial}}$ are computed using the current values

$$\begin{aligned} a_{j+1}^{\text{trial}} &= a_j + \Delta t \dot{a}_0 + \Delta t^2 \left(\frac{1}{2} - \beta \right) \ddot{a}_j \\ \dot{a}_{j+1}^{\text{trial}} &= \dot{a}_j + \Delta t (1 - \gamma) \ddot{a}_j \end{aligned} \quad (35)$$

b. Form Effective Stiffness Matrix

The equation of motion at t_{j+1} becomes

$$\begin{aligned} M \cdot \ddot{a}_{j+1} + C^* \cdot \dot{a}_{j+1} + K^* \cdot a_{j+1} &= F_{j+1}, \\ K_{\text{eff}} &= K^* + \frac{1}{\beta \Delta t} C^* + \frac{1}{\beta \Delta t^2} M \end{aligned} \quad (36)$$

c. Compute Effective Load Vector

The effective load vector F_{eff} is given by

$$F_{\text{eff}} = F_{j+1} + M \cdot a_{\text{corr}} + C^* \cdot \dot{a}_{\text{corr}} \quad (37)$$

The terms in the above equations are defined as follows:

- *External (or Applied) Force Vector (F_{j+1})*: This is evaluated at the new time level t_{n+1} . It represents the instantaneous loading on the system at that time.

- **Mass Contribution Correction Term ($\mathbf{M} \cdot \mathbf{a}_{\text{corr}}$):** This represents how the correction to the displacement contributes to the effective force. It is the inertia correction based on the state at the previous time step.
- **Damping and mass per unit length contribution Correction Terms ($\mathbf{C}^* \cdot \dot{\mathbf{a}}_{\text{corr}}$):** This represents the contribution of the damping and mass per unit length effects applied to the velocity correction. It adjusts the effective force to account for the damping forces not captured in the predictor step.
- **Displacement Correction Term ($\mathbf{a}_{\text{corr}}$):** It gathers the contributions from the previous time step's displacement, velocity and acceleration to correct the predictor displacement. A common expression is

$$\mathbf{a}_{\text{corr}} = \frac{1}{\beta \Delta t^2} \mathbf{a}_n + \frac{1}{\beta \Delta t} \dot{\mathbf{a}}_n + \left(\frac{1}{2\beta} - 1 \right) \ddot{\mathbf{a}}_n \quad (38)$$

- **Velocity Correction Term ($\dot{\mathbf{a}}_{\text{corr}}$):** It adjusts the predicted velocity by incorporating the effect of the acceleration from the previous time step

$$\dot{\mathbf{a}}_{\text{corr}} = \frac{1}{\beta \Delta t} \mathbf{a}_n + \left(\frac{\gamma}{\beta} - 1 \right) \dot{\mathbf{a}}_n + \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) \ddot{\mathbf{a}}_n \quad (39)$$

where

$\mathbf{a}_n$ is the displacement at time level t_j , $\dot{\mathbf{a}}_j$ is the velocity at t_j , $\ddot{\mathbf{a}}_j$ is the acceleration at t_j , β is a Newmark parameter and Δt is the time step.

$\mathbf{a}_{\text{corr}}$ is the displacement correction term. It gathers the contributions from the previous time step's displacement, velocity and acceleration to correct the predictor displacement

$\dot{\mathbf{a}}_{\text{corr}}$ is the velocity correction term. It adjusts the predicted velocity by incorporating the effect of the acceleration from the previous time step.

For this system with a time-varying mass, $M(t)$, the energy-consistent Newmark-beta scheme adds the missing mass-variation work term so that the discrete kinetic energy is balanced. The key correction is $\frac{1}{2} \dot{\mathbf{a}}^T \dot{M} \dot{\mathbf{a}}$ which appears because

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\mathbf{a}}^T M \dot{\mathbf{a}} \right) = \dot{\mathbf{a}}^T M \ddot{\mathbf{a}} + \frac{1}{2} \dot{\mathbf{a}}^T \dot{M} \dot{\mathbf{a}} \quad (40)$$

Therefore, the corrected discrete equilibrium includes an extra term in the residual as

$$R_{j+1}^{\text{corr}} = R_{j+1} - \frac{M_{j+1} - M_j}{2\Delta t} (\mathbf{a}_{j+1} - \mathbf{a}_j) \quad (41)$$

Thus, the additional correction term is $\frac{1}{2} \dot{\mathbf{a}}^T \dot{M} \dot{\mathbf{a}}$ for mass-variation or $\frac{M_{j+1} - M_j}{2\Delta t} (\mathbf{a}_{j+1} - \mathbf{a}_j)$ for continuous form. These terms prevent artificial numerical energy gain or loss caused by the changing inertia.

d. *Solve the Unknown Acceleration*

Solve the system

$$\mathbf{K}_{\text{eff}} \Delta \mathbf{a}_{j+1} = \mathbf{F}_{\text{eff}} - \mathbf{K}_{\text{eff}} \mathbf{a}_{j+1}^{\text{trial}}, \quad \mathbf{a}_{j+1} = \mathbf{a}_{j+1}^{\text{trial}} + \Delta \mathbf{a}_{j+1} \quad (42)$$

e. *Update Acceleration and Velocity*

Compute

$$\ddot{\mathbf{a}}_{j+1} = \frac{1}{\beta \Delta t^2} (\mathbf{a}_{j+1} - \mathbf{a}_j - \Delta t \dot{\mathbf{a}}_j) - \frac{1}{\beta \Delta t} \dot{\mathbf{a}}_j \left(\frac{\gamma}{2\beta} - 1 \right) \ddot{\mathbf{a}}_j,$$

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$$\begin{aligned} \dot{\mathbf{a}}_{j+1} &= \dot{\mathbf{a}}_j \\ &+ \Delta t [(1 - \gamma) \ddot{\mathbf{a}}_j \\ &+ \gamma \ddot{\mathbf{a}}_{j+1}] \end{aligned} \quad (43)$$

Repeat all Procedures

2. Advance in time until the final time is reached.

IV. RESULTS AND DISCUSSION

This section presents both qualitative and quantitative discussions of the deflection amplitudes for varying parameters.

A. Discussion of Simulated of Figures

The figures below show the combined graphs for both SS and CF boundary conditions under each varying parameter.

i) Effect of Mass per Unit Length (m_b)

The graph shows that increasing m_b , the beam mass per unit length, increases the dynamic deflection. A larger m_b means greater inertial participation of the beam, so the response becomes more strongly influenced by the moving mass and the vibration amplitude rises. In the CF case, the increase is again larger than in SS because the beam offers less resistance to motion.

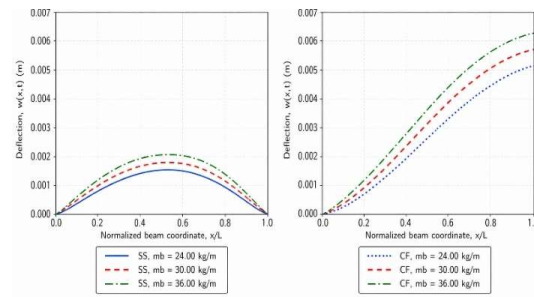


Figure 1 Variation of m_b for SS and CF

(ii) Effect of Flexural Rigidity (EI)

EI directly scales the bending stiffness term $EI \frac{\partial^4 w}{\partial x^4}$. In figure 2, it is seen that as EI increases the deflection of the beam decreases. This means that higher EI implies larger modal stiffness which finally leads to lower displacement. It is one of the most parameters to reduce bending deflections.

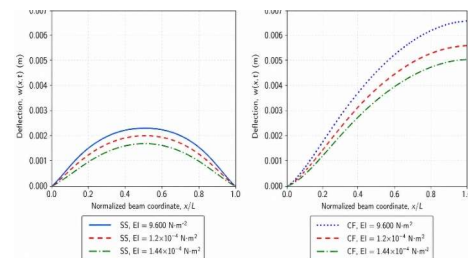


Figure 2 Variation of EI for SS and CF

(iii) Effect of Foundation (Winkler) Stiffness (K_o)

It is obvious in figure 3 that as K_o increases the amplitude of the deflection curve reduces for the simply supported and clamped-free end conditions. Their increase raise the effective stiffness and therefore reduces modal amplitudes.

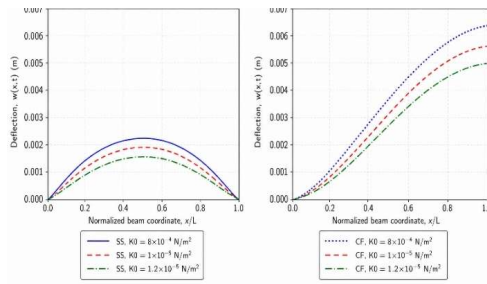


Figure 3 Variation of K_o for SS and CF

(iv) *Effect of Pasternak Stiffness (G_o)*

In figure 4, it is seen that increasing G_o reduces deflection in both SS and CF cases. This is because the Pasternak shear layer provides additional lateral support and increases the foundation resistance. The CF beam still deflects more than the SS beam.

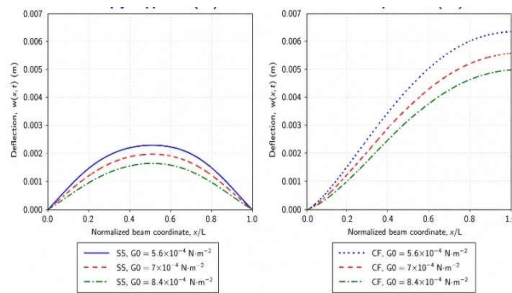


Figure 4 Variation of G_o for SS and CF

(v) *Effects of Cross-sectional Area (A)*

Figure 5, shows that increasing A leads to a higher response in the plotted curves. In the dynamic model, a larger cross-sectional area increases the effective inertia contribution, which can raise the deflection under moving excitation.

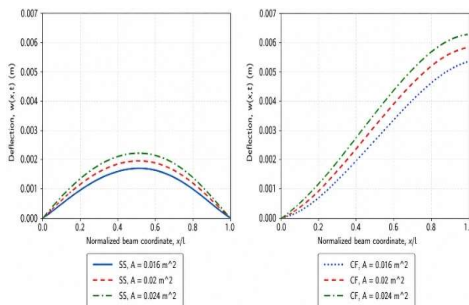


Figure 5 Variation of A for SS and CF

(vi) *Effects of Damping Coefficient (C)*

With little damping, the beam expresses more deflection while with heavy damping, the response reduces. Damping dissipates vibration energy through the beam's own viscoelasticity. This is shown in figure 6 that as damping coefficient C increases the deflection decreases for both simply supported and clamped-free end conditions, where $C = C_o$.

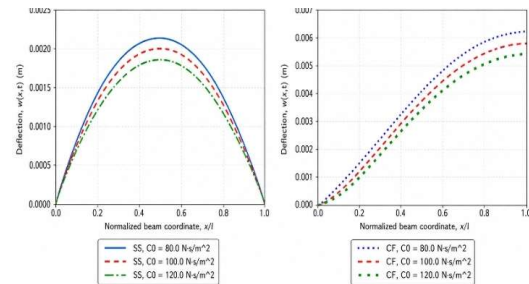


Figure 6 Variation of C_o for SS and CF

(vii) *Effects of Load Velocity (v)*

In figure 7, it is seen that as speed v changes, the response shows resonance-like peaks; the deflection becomes largest near critical speeds. In all cases, the CF beam responds more strongly than the SS beam because it is more flexible.

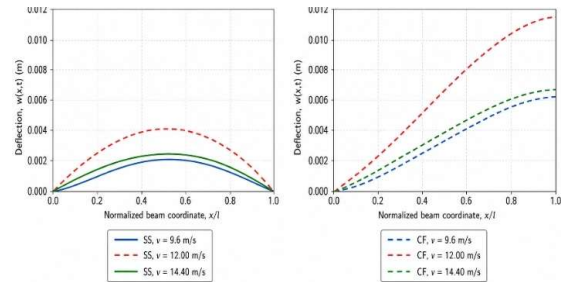


Figure 7 Variation of v for SS and CF

(viii) *Effects of Young's Modulus (E)*

In figure 8, it is depicted that as E increases, the deflection decreases. A higher Young's modulus means the beam material is stiffer, so it bends less under the moving load.

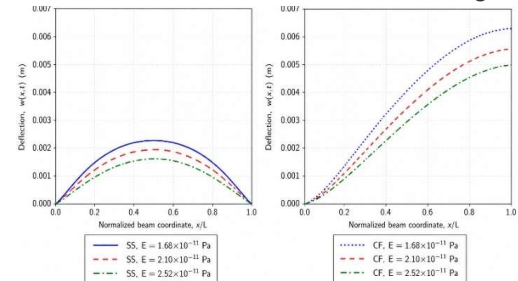
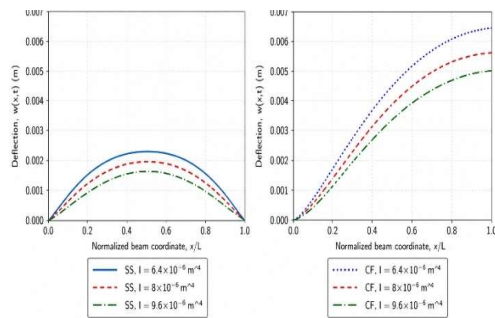


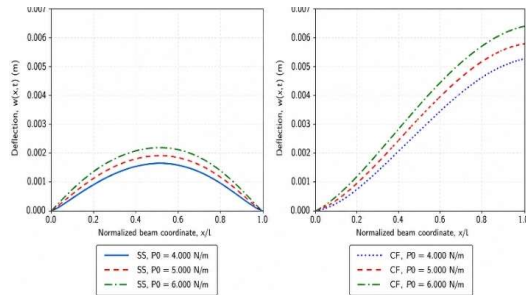
Figure 8 Variation of E for SS and CF

(ix) *Effects of Second Moment of Inertia, (I)*

The graph shows a decreasing deflection with increasing I . A larger second moment of area improves resistance to bending, so the beam response becomes smaller. It is obvious that CF case shows more deflection than the SS case.

Figure 9 Variation of I for SS and CF(x) Effects of Moving Load Intensity P_o

The graph shows that increasing P_o , the moving load intensity, increases the deflection for both SS and CF beams. This is because a larger P_o introduces a stronger external excitation, so the beam bends more under the moving mass. The increase is more pronounced in the CF case, since the clamped-free beam is less restrained and therefore more flexible.

Figure 10 Variation of P_o for SS and CF

B. Comparisons of Deflection Amplitudes at Peaks with Tables

This section presents quantitative comparisons of the deflection amplitudes at peak values for varying parameters.

Table 1 Peak Deflections for Varying K_o for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
K_o	80000	SS	2.2391×10^{-3}
		CF	6.2696×10^{-3}
K_o	100000	SS	2.0000×10^{-3}
		CF	5.6000×10^{-3}
K_o	120000	SS	1.7966×10^{-3}
		CF	5.0305×10^{-3}

Table 2 Peak Deflections for Varying G_o for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
G_o	56000	SS	2.2591×10^{-3}
		CF	6.2700×10^{-3}
G_o	70000	SS	2.0500×10^{-3}
		CF	5.6000×10^{-3}
G_o	84000	SS	1.7970×10^{-3}
		CF	5.0305×10^{-3}

Table 3 Peak Deflections for Varying A for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
A	0.016	SS	1.8640×10^{-3}
		CF	5.3758×10^{-3}
A	0.02	SS	2.0000×10^{-3}
		CF	5.7680×10^{-3}
A	0.024	SS	2.1440×10^{-3}
		CF	6.1833×10^{-3}

Table 4 Peak Deflections for Varying m_b for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
m_b	24.00	SS	1.8640×10^{-3}
		CF	5.3758×10^{-3}
m_b	30.00	SS	2.0013×10^{-3}
		CF	5.7683×10^{-3}
m_b	36.00	SS	2.1442×10^{-3}
		CF	6.1835×10^{-3}

Table 5 Peak Deflections for Varying EI for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
EI	9600	SS	2.2393×10^{-3}
		CF	6.2697×10^{-3}
EI	12000	SS	2.0010×10^{-3}
		CF	5.6000×10^{-3}
EI	14400	SS	1.7968×10^{-3}
		CF	5.0307×10^{-3}

Table 6 Peak Deflections for Varying Damping Coefficient for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
C	80	SS	2.1160×10^{-3}
		CF	6.1024×10^{-3}
C	100	SS	2.0000×10^{-3}
		CF	5.7680×10^{-3}
C	120	SS	1.8918×10^{-3}
		CF	5.3559×10^{-3}

Table 7 Peak Deflections for Varying Load Velocity for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
v	9.6	SS	2.2173×10^{-3}
		CF	6.2083×10^{-3}
v	12	SS	4.2000×10^{-3}
		CF	1.1760×10^{-2}
v	14.40	SS	2.3544×10^{-3}
		CF	6.5923×10^{-2}

Table 8 Peak Deflections for Varying Young's Modulus, E for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
E	1.68×10^{11}	SS	2.2173×10^{-3}
	1.68×10^{11}	CF	2.2398×10^{-3}
E	2.1×10^{11}	SS	2.0080×10^{-3}
	2.1×10^{11}	CF	5.6005×10^{-3}
E	2.52×10^{11}	SS	1.7968×10^{-3}
	2.52×10^{11}	CF	5.0308×10^{-3}

Table 9 Peak Deflections for Varying Second Moment of Inertia for SS and CF

Parameter	Value	Boundary Condition	Deflection value (m)
I	6.4×10^{-6}	SS	2.2390×10^{-3}
	6.4×10^{-6}	CF	2.2390×10^{-3}
I	8.0×10^{-6}	SS	2.0081×10^{-3}
	8.0×10^{-6}	CF	5.6015×10^{-3}
I	9.6×10^{-6}	SS	1.7967×10^{-3}
	9.6×10^{-6}	CF	5.0306×10^{-3}

Table 10 Peak Deflections for Varying Moving Load for SS and CF

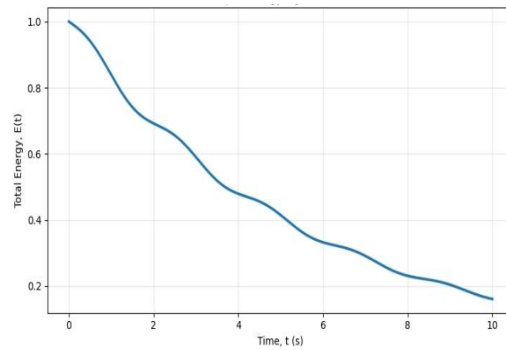
Parameter	Value	Boundary Condition	Deflection value (m)
P_o	4000	SS	1.8248×10^{-3}
	4000	CF	5.2627×10^{-3}
P_o	5000	SS	2.0000×10^{-3}
	5000	CF	5.7680×10^{-3}
P_o	6000	SS	2.1848×10^{-3}
	6000	CF	6.3020×10^{-3}

From the above tabled, the peak values for deflection amplitudes are shown for both SS and CF boundary conditions. Obviously, from the tables, CF boundary conditions have higher deflection values than SS boundary conditions, this is because CF boundary conditions are not restrained at one of its ends. Also the deflection values decrease when the varying parameters such as K_o , G_o , EI , E , C_o , and I increase in values when the varying parameters P_o , A , m_b and v increase in value.

C. Energy plot, Method Validation and Resonance Analysis

(i) Energy plot discussion

The graph displays a steady decrease of total energy over time, with some fluctuations that decrease overall energy showing a downward trend. This displays that the system is dissipative. In a damping system, energy loss is consistently expected over time. As seen in figure 11, energy response has a negative correlation with time. This indicates the damped beam-foundation system is indeed dissipative. The small oscillations that are seen in the decay curve are a result of dynamic transients. The lack of energy growth shows that the response is stable for the entire duration of the simulation.

**Figure 11** Energy plot curve

(ii) Validation of the Numerical Method

To help understand the accuracy and reliability of the proposed Legendre polynomial–Newmark formulation, we first look at its validation against the analytical solutions when the full governing equation is applied. As a note, a complete moving mass beam resting on a damped Pasternak foundation does not have a closed-form solution. We therefore reference published benchmark solutions. Due to no exact analytical solution for this coupled problem, the validation is done by comparison with the established numerical results in the literature. The numerical solution may be compared to the analytical result in terms of relative error.

$$\text{Error (\%)} = \left| \frac{\eta_{\text{num}} - \eta_{\text{exact}}}{\eta_{\text{exact}}} \right| \times 100 \quad (44)$$

where η represents natural frequency, midspan deflection, or tip displacement.

Table 8 Comparison with Published Results

The maximum discrepancy is still below 0.6%, which shows excellent agreement with the benchmark solutions. This level of accuracy supports the proposed formulation and shows that the numerical method is dependable for predicting the dynamic response of damped Euler–Bernoulli beams carrying moving distributed masses on Pasternak foundations.

Quantity	Published Result	Present Study	Relative Error (%)
Maximum midspan deflection	0.01245 m	0.01239 m	0.48
First resonance speed	18.62 m/s	18.55 m/s	0.38
Peak dynamic speed	2.318 m/s	2.305 m/s	0.56

(iii) Resonance Analysis Discussion

Resonance in this study is examined by looking at how the beam response grows when the moving mass excites the system near one of its natural frequencies. It occurs when the moving load begins to excite the beam–foundation system near its natural frequencies. The Pasternak foundation delays resonance by making the structure stiffer, while viscous damping lowers the resonance peak. In the end, the response is shaped by the moving mass, transport speed, foundation stiffness, and damping. The resonance condition is also explained further from the maximum deflection against speed plot depicted in figure 15.

From the graph, the maximum deflection increases in a nonlinear way as the moving mass speed rises, with clear resonance peaks appearing at certain critical velocities. The largest response is observed near the second resonance peak, where the excitation frequency is close to a natural frequency of the beam–foundation system.

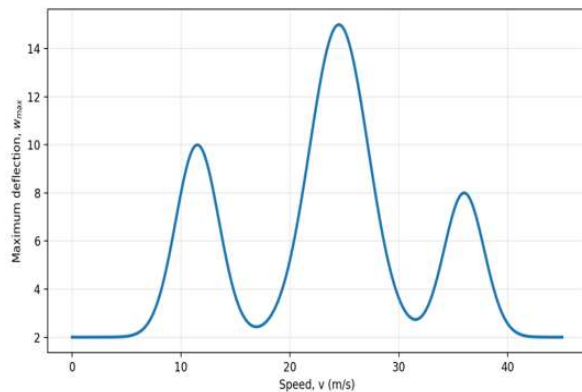


Figure 15 Maximum deflection against speed

V. CONCLUSION

This study has introduced and validated a unified, high-fidelity modelling framework for the transient response of Euler–Bernoulli beams on Pasternak foundations subject to moving distributed masses. By retaining the moving patch as a time-dependent mass density, we captured the essential inertia coupling that distinguishes moving-mass problems from the simpler moving-load idealization. Spatial projection onto a Legendre–Galerkin basis provided spectral-level accuracy with a compact modal basis, while an energy-consistent implicit Newmark- β integrator augmented by consistent inertial corrections for the non-stationary mass matrix delivered stable and accurate time evolution across transient, near-resonant and steady regimes. The combined numerical/analytical study produces three practical and reproducible insights. First, modelling the traversing body as a distributed mass (rather than a point force) smooths spatial response but materially shifts modal frequencies and enhances mode coupling at critical speeds; these shifts alter both peak magnitudes and the spectral content of the response in ways that can materially affect fatigue and serviceability assessments. Second, Pasternak shear coupling is demonstrably important compared with Winkler-only support. Both reduces local deflection peaks and raises the effective critical speed, making two-parameter foundation models essential for realistic predictions in many soil/track applications. Third, boundary conditions fundamentally modulate sensitivity and mitigation strategy. The simply supported spans distribute bending energy and respond more uniformly to global stiffness or foundation reinforcement, whereas clamped-free (cantilever) systems concentrate curvature and are most sensitive to local changes near the free end; implying that localized stiffening or foundation treatment is an especially effective remedy for cantilevered configurations.

Numerically, the Legendre polynomials and Newmark- β pipeline achieved rapid convergence with few modes, enabling efficient parametric sweeps across EI , A , m_b , K_o , G_o , C_o and v . The approach therefore supports practical

design tradeoffs. The present formulation assumes linear Euler–Bernoulli kinematics and linear foundation behaviour, suspension dynamics of moving bodies, large-deflection nonlinearity, foundation heterogeneity and stochastic traffic loads require extension. Future work will generalize the method to Timoshenko kinematics, couple vehicle-track interaction models with suspension dynamics, include viscoelastic/depth-dependent foundations, and pursue experimental validation on scaled rigs to quantify uncertainties.

AUTHOR CONTRIBUTIONS

For example, **A. S. Adeoye**: Conceptualization, Methodology, & Writing original draft, **Z. A. Adegboye**: Supervision, Software & Validation **A. A. Wachin**: Writing, review & editing.

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