Stability Framework for Quintuple Fixed Point of Mixed Monotone Operators with Contractive Conditions in Cauchy Spaces

S. A. Aniki[1, *], P. B. Metiboba[2]

[1]Department of Mathematics and Statistics, Faculty of Science, Confluence University of Science and Technology, Osara, Kogi State, Nigeria.
[2]Department of Computer Science, Faculty of Computing and Informatics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria.

ABSTRACT: The stability of quintuple fixed point for mixed monotone operators in partially ordered metric spaces remains underexplored, especially under generalised contractive conditions in Cauchy spaces. This study addresses this theoretical gap by developing a novel iterative procedure for quintuple fixed point. The methodology introduces a generalised contractive type mapping and uses recursive inequalities alongside matrix-based comparison techniques to establish convergence and stability criteria. These analytical tools are adapted to vector-valued sequences, enabling precise control over componentwise convergence behaviour. A new iteration framework is defined and rigorously analysed within a complete Cauchy-type metric space. The results reveal that the scheme ensures both the existence and stability of the fixed point. An illustrative real-valued application validates the theoretical framework. This approach unifies and extends previous work on coupled, tripled, and quadrupled fixed point, offering a robust analytical foundation for high-dimensional operator systems and future applications in nonlinear, fractional, or fuzzy dynamical models.

KEYWORDS: Quintuple fixed point, Mixed monotone operators, Stability analysis, Contractive conditions, Cauchy spaces.

[Received May 25, 2025; Revised July 3, 2025; Accepted Aug. 30, 2025] Print ISSN: 0189-9546 | Online ISSN: 2437-2110

I. INTRODUCTION

Fixed point theory is a cornerstone of nonlinear analysis with profound implications for differential equations, dynamic systems, and applied mathematics. From Banach's seminal work to contemporary explorations of multidimensional mappings, this field's evolution has witnessed an increasingly intricate structural understanding of operators in various topological and metric spaces (Mannan *et al.*, 2021). Studying fixed points in partially ordered metric spaces has generated considerable interest, particularly for mappings exhibiting monotone behaviour. Lakshmikantham and Ciric (2009) provided one of the earliest explorations of coupled fixed points under such settings, which has since motivated numerous extensions to tripled, quadrupled, and even quintupled configurations. For instance, the works of Gupta and Manro (2017), Berinde and Pacurar (2017), and Deshpande and Handa (2015) explore the structural technicalities of higher-order fixed points under mixed monotone conditions.

Recent developments in the field have focused on contractive conditions, either of integral type (Khan *et al.*, 2020) or in generalised forms (Ravibabu *et al.*, 2018; Radenovic, 2014), to ensure the existence, uniqueness and stability of fixed points. Rehman *et al.* (2019) and Timis (2014) have analysed tripled fixed point iterations, particularly emphasising their stability properties. The notion of stability in fixed point theory, interpreted here as the sensitivity of fixed point sequences to perturbations in initial conditions, has proven particularly successful. Aniki and Rauf (2019), Aniki (2021, 2022), and Rauf and Aniki (2021) have contributed significantly to this domain by establishing iterative procedures and stability analyses in both standard and Cauchy-type metric spaces. These efforts reflect a broader trend toward the incorporation of algebraic and topological structures, such as C*-algebra-valued fuzzy metric spaces (Aniki, 2022), and the refinement of contraction mappings using matrix inequalities and vector analysis (Imdad *et al.*, 2021; Hammad, 2024).

More complex operator frameworks, such as those introduced by Karapinar and Van Luong (2012) and Algolam *et al.* (2024), leverage fractional differential equations and multidimensional metric constructs to broaden the traditional boundaries of fixed point theory. These contributions emphasise the adaptability of fixed point methodologies in capturing nonlinear dynamics across high-dimensional and non-Euclidean settings. Likewise, the works of Ali et al. (2025) and Hammad and Kattan (2025) have utilised the Ran-Reurings theorem and novel F-contractive mappings to effectively address tripled and strongly nonlinear fixed point systems, particularly within partially ordered spaces. Extending this trend, Aniki and Rauf (2020) developed a

*Corresponding author: anikisa@custech.edu.ng
doi: http://doi.org/10.63746/njtd.v22i4.3694

comprehensive stability system for quadrupled fixed points under contractive-type operators, highlighting the utility of recursive inequality techniques and metric completeness in analysing convergence.

Given this rich landscape, the present study proposes a novel approach for the analysis of quintuple fixed points arising from mixed monotone operators defined over partially ordered Cauchy spaces. Our framework generalises the structural assumptions underlying previous works such as those by Gregorio (2019), Kadelburg and Radenovic (2015), and Roldan *et al.* (2014), while introducing an iterative scheme that meets enhanced stability criteria. Moreover, we demonstrate that under suitable contractive conditions, akin to those proposed by Saddeek and Ahmed (2024), the quintuple fixed point iteration exhibits convergence and stability, even in complex multidimensional domains. This paper also refines and extends the analytical toolkit used to establish fixed point results. We draw upon recursive inequality methods similar to those in the works of Rauf, Aiyetan, and Aniki (2017) and matrix-theoretic convergence analysis inspired by the results of Agarwal, Jleli, and Samet (2018). These innovations climax in the main theorem of this study, which provides a strong existence and stability criterion for quintuple fixed points.

In essence, given the progressive development of fixed point theory from coupled to tripled and quadrupled configurations, as seen in recent literature, it becomes imperative to investigate the structural and stability properties of even higher-order mappings. The analysis of quintuple fixed points provides a logical and non-trivial extension of existing results, especially in the context of partially ordered metric spaces equipped with mixed monotone operators. While earlier studies have established the foundational stability of iterative schemes under contractive conditions for lower-dimensional cases, there remains a significant theoretical gap concerning the convergence behaviour and robustness of iterative procedures in quintupled frameworks. Motivated by this lacuna, the present work develops a generalised contractive-type condition and formulates a new iteration scheme for quintuple fixed points in Cauchy-type metric spaces. Our results extend and unify prior stability theorems and also provide a broader analytical system for handling multidimensional nonlinear operator equations in both pure and applied mathematical contexts.

II. MATERIALS AND METHODS.

To facilitate the presentation of our main results, we introduce some notations that will be essential throughout the analysis.

Let $(X,\leq)$ be a partially ordered set and let d be a metric on X such that (X,d) forms a complete metric space. Consider the Cartesian product X^5 which is endowed with a partial order defined as follows:

$$(m,n,p,q,r) \leq (s,t,u,v,w) \Leftrightarrow s \geq m, t \leq n, u \geq p, v \leq q, w \geq r, x \leq s \quad \forall\ (m,n,p,q,r),(s,t,u,v,w) \in X^5. \tag{1}$$

Definition 1. Let $(X,\leq)$ be a partially ordered set and let $T: X^5 \to X$ be a mapping. The function T is said to possess the mixed monotone property if it is monotone nondecreasing with respect to the variables s, u and w, and monotone nonincreasing with respect to the variables t and v. That is, for all $s,t,u,v,w \in X$

$$s_1, s_2 \in X, s_1 \leq s_2 \Rightarrow T(s_1,t,u,v,w) \leq T(s_2,t,u,v,w), \tag{2}$$
$$t_1, t_2 \in X, t_1 \leq t_2 \Rightarrow T(s,t_1,u,v,w) \geq T(s,t_2,u,v,w), \tag{3}$$
$$u_1, u_2 \in X, u_1 \leq u_2 \Rightarrow T(s,t,u_1,v,w) \leq T(s,t,u_2,v,w), \tag{4}$$
$$v_1, v_2 \in X, v_1 \leq v_2 \Rightarrow T(s,t,u,v_1,w) \geq T(s,t,u,v_2,w), \tag{5}$$
and
$$w_1, w_2 \in X, w_1 \leq w_2 \Rightarrow T(s,t,u,v,w_1) \leq T(s,t,u,v,w_2). \tag{6}$$

Definition 2. An element $(s,t,u,v,w) \in X^5$ is called a quintupled fixed point of the mapping $T: X^5 \to X$, if $T(s,t,u,v,w) = s, T(t,s,t,v,w) = t, T(u,s,t,u,w) = u, T(v,v,t,u,v) = v, T(w,v,u,t,s) = w$.

Definition 3. The mapping $T: X^5 \to X$ is said to be $(\vartheta,\kappa,\pi,\lambda,\mu)$ −contraction if and only if there exist constants $\vartheta \geq 0, \kappa \geq 0, \pi \geq, \lambda \geq 0, \mu \geq 0, \vartheta+\kappa+\pi+\lambda+\mu < 1$, such that $\forall\ s,t,u,v,w,m,n,p,q,r \in X$,

$$d[T(s,t,u,v,w),T(m,n,p,q,r)] = \vartheta d(s,m) + \kappa d(t,n) + \pi d(u,p) + \lambda d(v,q) + \mu d(w,r), \tag{7}$$

let $X, Y \in M_{(m,n)}(\mathbb{R})$ be two matrices. Then, $X \leq Y$; if $\alpha_{ij} \leq \beta_{ij}\ \forall\ i = \overline{1,m}, j = \overline{1,n}$.

The following lemma will play a crucial role in establishing our main result on stability.

Lemma 1. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of nonnegative real numbers and let h be a constant such that $0 \leq h \leq 1$. Suppose the sequences satisfy the recursive inequality

$$\alpha_{n+1} \leq h\alpha_n + \beta_n, \quad n \geq 0,$$

If $\lim_{n\to\infty} \beta_n = 0$, then it follows that $\lim_{n\to\infty} \alpha_n = 0$.

The subsequent result extends this lemma to the setting of vector-valued sequences, where inequalities between vectors are interpreted componentwise.

Lemma 2. Let $\{m_n\},\{n_n\}\{p_n\},\{q_n\},\{r_n\}$ be sequences of nonnegative real numbers. Consider a matrix $X \in M_{(5,5)}(\mathbb{R})$ with nonnegative elements, such that

$$\begin{pmatrix} m_{n+1} \\ n_{n+1} \\ p_{n+1} \\ q_{n+1} \\ r_{n+1} \end{pmatrix} \leq X.\begin{pmatrix} m_n \\ n_n \\ p_n \\ q_n \\ r_n \end{pmatrix} + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{pmatrix}, \quad n \geq 0, \tag{8}$$

with

i. $\lim_{n\to\infty} X^n = 0_5$,

ii. $\sum_{k=0}^{\infty} \eta_k < \infty$, $\sum_{k=0}^{\infty} \varepsilon_k < \infty$, $\sum_{k=0}^{\infty} \xi_k < \infty$, $\sum_{k=0}^{\infty} \delta_k < \infty$, and $\sum_{k=0}^{\infty} \gamma_k < \infty$.

If $\lim_{n\to\infty} \begin{pmatrix} m_n \\ n_n \\ p_n \\ q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, then $\lim_{n\to\infty} \begin{pmatrix} m_n \\ n_n \\ p_n \\ q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Proof. For $X = 0 \in M_{(5,5)}$, the conclusion is obvious. By setting $n = k$ in equation (8) and summing the resulting inequalities for $k = 0,1,2,\dots,n$, we obtain the following expression valid for all $k = 0,1,2,\dots,n$.

$$\begin{pmatrix} m_{n+1} \\ n_{n+1} \\ p_{n+1} \\ q_{n+1} \\ r_{n+1} \end{pmatrix} \le X^{n+1} \cdot \begin{pmatrix} m_0 \\ n_0 \\ p_0 \\ q_0 \\ r_0 \end{pmatrix} + \sum_{k=0}^{n} X^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \xi_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix}, \quad n \ge 0, \quad (9)$$

From condition (ii) of Lemma 2, it follows that the sequences of partial sums $\{\mathrm{H}_n\}, \{\mathrm{E}_n\}, \{\Xi_n\}, \{\Delta_n\}, \{\Gamma_n\}$, are given respectively by $\mathrm{H}_n = \eta_0 + \eta_1 + \dots + \eta_n$, $\mathrm{E}_n = \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_n$, $\Xi_n = \xi_0 + \xi_1 + \dots + \xi_n$, $\Delta_n = \delta_0 + \delta_1 + \dots + \delta_n$, and$\Gamma_n = \gamma_0 + \gamma_1 + \dots + \gamma_n$, for $n \ge 0$, converge respectively to some $\mathrm{H}_n \ge 0$, $\mathrm{E}_n \ge 0$, $\Xi_n \ge 0$, $\Delta_n \ge 0$ and $\Gamma_n \ge 0$ and hence, they are bounded.

Let $M > 0$ be such that

$$\begin{pmatrix} \mathrm{H}_n \\ \mathrm{E}_n \\ \Xi_n \\ \Delta_n \\ \Gamma_n \end{pmatrix} \le M \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall\ n \ge 0. \quad (10)$$

By condition (i) of Lemma 2, for every $e > 0$, there exists an integer $N = N(e)$ such that $X^n \le \frac{e}{2M} . I_5, \forall\, n \ge N,\ M > 0$.

Write

$$\sum_{k=0}^{n} X^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \xi_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = X^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \xi_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \dots + X^N \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \xi_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} + X^{N-1} \begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \xi_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \dots + I_5 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{pmatrix} \quad (11)$$

But

$$X^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \xi_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \dots + X^N \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \xi_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \le \frac{e}{2M} \cdot I_5 \left[\begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \xi_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \dots + \begin{pmatrix} \eta_{n-N} \\ \varepsilon_{n-N} \\ \xi_{n-N} \\ \delta_{n-N} \\ \gamma_{n-N} \end{pmatrix} \right]$$

$$= \frac{e}{2M} \cdot I_5 \begin{pmatrix} \mathrm{H}_{n-N} \\ \mathrm{E}_{n-N} \\ \Xi_{n-N} \\ \Delta_{n-N} \\ \Gamma_{n-N} \end{pmatrix} \le \frac{e}{2M} \cdot I_5 \cdot M \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{e}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall\ n \ge N. \quad (12)$$

On the other hand, let $X' = max\{I_5, X, \dots, X^{N-1}\}$. Then, the following result is obtained.

$$X^{N-1} \begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \xi_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \dots + I_5 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{pmatrix} \le X' \left[\begin{pmatrix} \eta_{n-N+1} \\ \varepsilon_{n-N+1} \\ \xi_{n-N+1} \\ \delta_{n-N+1} \\ \gamma_{n-N+1} \end{pmatrix} + \dots + \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{pmatrix} \right] = X' \begin{pmatrix} \mathrm{H}_n - \mathrm{H}_{n-N} \\ \mathrm{E}_n - \mathrm{E}_{n-N} \\ \Xi_n - \Xi_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix}. \quad (13)$$

Since N is fixed, then $\lim_{n\to\infty} \mathrm{H}_n = \lim_{n\to\infty} \mathrm{H}_{n-N} = \mathrm{H}$, $\lim_{n\to\infty} \mathrm{E}_n = \lim_{n\to\infty} \mathrm{E}_{n-N} = \mathrm{E}$, $\lim_{n\to\infty} \Xi_n = \lim_{n\to\infty} \Xi_{n-N} = \Xi$, $\lim_{n\to\infty} \Delta_n = \lim_{n\to\infty} \Delta_{n-N} = \Delta$, $\lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \Gamma_{n-N} = \Gamma$, which shows that there exists a positive integer k such that

$$X' \begin{pmatrix} \mathrm{H}_n - \mathrm{H}_{n-N} \\ \mathrm{E}_n - \mathrm{E}_{n-N} \\ \Xi_n - \Xi_{n-N} \\ \Delta_n - \Delta_{n-N} \\ \Gamma_n - \Gamma_{n-N} \end{pmatrix} < \frac{e}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \forall\ n \ge k. \quad (14)$$

Now, for $m = max\{k, N\}$, the following is gotten

$$X^n \begin{pmatrix} \eta_0 \\ \varepsilon_0 \\ \xi_0 \\ \delta_0 \\ \gamma_0 \end{pmatrix} + \dots + I_5 \begin{pmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{pmatrix} < e \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \forall\, n \ge m,$$

and therefore, $\lim_{n\to\infty} \sum_{k=0}^{n} X^k \begin{pmatrix} \eta_{n-k} \\ \varepsilon_{n-k} \\ \xi_{n-k} \\ \delta_{n-k} \\ \gamma_{n-k} \end{pmatrix} = 0.$

Taking the limit in equation (9) and using the fact that $\lim_{n\to\infty} X^n = 0$, it follows that

$$\lim_{n\to\infty} \begin{pmatrix} m_n \\ n_n \\ p_n \\ q_n \\ r_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (15)$$

as required.

Definition 4. Let (X, d) be a metric space, and let $T: X^5 \to X$ be a mapping. For a given initial point $(s_0, t_0, u_0, v_0, w_0) \in X^5$, consider the sequence $\{(s_n, t_n, u_n, v_n, w_n)\} \subset X^5$ defined recursively by

$$s_{n+1} = T(s_n, t_n, u_n, v_n, w_n),\ t_{n+1} = T(t_n, s_n, t_n, v_n, w_n),\ u_{n+1} = T(u_n, s_n, t_n, u_n, w_n),\ v_{n+1} = T(v_n, v_n, t_n, u_n, v_n),\ w_{n+1} = T(w_n, v_n, u_n, t_n, s_n) \quad (16)$$

For all $n = 0,1,2,\dots$. This iterative process is referred to as the quintupled fixed point iteration scheme.

Definition 5. Let (X, d) be a complete metric space, and define the set of quintupled fixed points of the mapping $T: X^5 \to X$ by

$$Fix_t(T) = \{(s^*, t^*, u^*, v^*, w^*) \in X^5 /\ T(s^*, t^*, u^*, v^*, w^*) = s^*,\ T(t^*, s^*, t^*, v^*, w^*) = t^*,\ T(u^*, s^*, t^*, u^*, w^*) =$$

$$u^*,\ T(v^*,v^*,t^*,u^*,v^*)=v^*,\ T(w^*,v^*,u^*,t^*,s^*)=w^*\}. \quad (17)$$

This set, $Fix_t(T)$, consists of all quintuplets in X^5 that satisfy the fixed point conditions defined componentwise under the action of T.

Definition 6. Let $\{(m_n,n_n,p_n,q_n,r_n)\}\subset X^5$ be an arbitrary sequence. For each $n=0,1,2,\ldots$, define the following quantities:

$$\eta_n=d\big(m_{n+1},T(m_n,n_n,p_n,q_n,r_n)\big),\ \varepsilon_n=d\big(n_{n+1},T(n_n,m_n,n_n,q_n,r_n)\big),\ \xi_n=d\big(p_{n+1},T(p_n,m_n,n_n,p_n,r_n)\big),\ \delta_n=d\big(q_{n+1},T(q_n,q_n,n_n,p_n,q_n)\big),\ \gamma_n=d\big(r_{n+1},T(r_n,q_n,p_n,n_n,m_n)\big). \quad (18)$$

The iterative process defined in equation (16) is said to be T −stable (or stable with respect to T) if and only if $\lim_{n\to\infty}(\eta_n,\varepsilon_n,\xi_n,\delta_n,\gamma_n)=0_{\mathbb{R}^5}$ implies that $\lim_{n\to\infty}(m_n,n_n,p_n,q_n,r_n)=(s^*,t^*,u^*,v^*,w^*)$, where (s^*,t^*,u^*,v^*,w^*) is a quintupled fixed point of the mapping T.

III. MAIN RESULTS

Theorem 1. Let $(X,\leq)$ be a partially ordered set equipped with a metric d such that (X,d) forms a complete metric space. Consider a continuous mapping $T:X^5\to X$ that possesses the mixed monotone property on X, and suppose it satisfies a specified functional equation denoted as equation (7). Assume there exist initial points $s_0,t_0,u_0,v_0,w_0\in X$ satisfying the following inequalities:

$$s_0\leq T(s_0,t_0,u_0,v_0,w_0),$$

$$t_0\geq T(t_0,s_0,t_0,v_0,w_0),$$

$$u_0\leq T(u_0,s_0,t_0,u_0,w_0),$$

$$v_0\geq T(v_0,v_0,t_0,u_0,v_0),$$

$$w_0\leq T(w_0,v_0,u_0,t_0,s_0).$$

Then, there exist $s^*,t^*,u^*,v^*,w^*\in X$ such that each satisfies the following fixed point relations:

$$s^*=T(s^*,t^*,u^*,v^*,w^*),$$

$$t^*=T(t^*,s^*,t^*,v^*,w^*),$$

$$u^*=T(u^*,s^*,t^*,u^*,w^*),$$

$$v^*=T(v^*,v^*,t^*,u^*,v^*),$$

$$w^*=T(w^*,v^*,u^*,t^*,s^*).$$

For every $(s,t,u,v,w),(s_1,t_1,u_1,v_1,w_1)\in X^5$, there exist $(m,n,p,q,r)\in X^5$ that is comparable to (s,t,u,v,w) and (s_1,t_1,u_1,v_1,w_1). Let $(s_0,t_0,u_0,v_0,w_0)\in X^5$, be given, and consider the sequence $\{(s_n,t_n,u_n,v_n,w_n)\}\subset X^5$ defined by the iterative process (16). Then, this iterative procedure converges and is stable for the mapping T, ensuring the existence and stability of the quintupled fixed point.

Proof. Let $\{(s_n,t_n,u_n,v_n,w_n)\}\subset X^5$,

$$\eta_n=d\big(m_{n+1},T(m_n,n_n,p_n,q_n,r_n)\big),$$
$$\varepsilon_n=d\big(n_{n+1},T(n_n,m_n,n_n,q_n,r_n)\big),$$
$$\xi_n=d\big(p_{n+1},T(p_n,m_n,n_n,p_n,r_n)\big),$$
$$\delta_n=d\big(q_{n+1},T(q_n,q_n,n_n,p_n,v_n)\big),$$

and

$$\gamma_n=d\big(r_{n+1},T(r_n,q_n,p_n,n_n,m_n)\big).$$

Assume also that $\lim_{n\to\infty}\eta_n=\lim_{n\to\infty}\varepsilon_n=\lim_{n\to\infty}\xi_n=\lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\gamma_n=0$, to establish that $\lim_{n\to\infty}m_n=s^*$, $\lim_{n\to\infty}n_n=t^*$, $\lim_{n\to\infty}p_n=u^*$, $\lim_{n\to\infty}q_n=v^*$ and $\lim_{n\to\infty}r_n=w^*$.

Hence, by applying the $(\vartheta,\kappa,\pi,\lambda,\mu)$ −contraction condition as stated in equation (7), we derive the following result:

$$\begin{aligned}d(m_{n+1},s^*)&\leq d\big(m_{n+1},T(m_n,n_n,p_n,q_n,r_n)\big)+d\big(T(m_n,n_n,p_n,q_n,r_n),s^*\big)\\&=d\big(T(m_n,n_n,p_n,q_n,r_n),T(s^*,t^*,u^*,v^*,w^*)\big)+\eta_n\\&\leq\vartheta d(m_n,s^*)+\kappa d(n_n,t^*)+\pi d(p_n,u^*)+\lambda d(q_n,v^*)+\mu d(r_n,w^*)+\eta_n,\end{aligned} \quad (19)$$

$$\begin{aligned}d(n_{n+1},t^*)&\leq d\big(n_{n+1},T(n_n,m_n,n_n,q_n,r_n)\big)+d\big(T(n_n,m_n,n_n,q_n,r_n),t^*\big)\\&=d\big(T(n_n,m_n,n_n,q_n,r_n),T(t^*,s^*,t^*,v^*,w^*)\big)+\varepsilon_n\\&\leq\vartheta d(n_n,t^*)+\kappa d(m_n,s^*)+\pi d(n_n,t^*)+\lambda d(q_n,v^*)+\mu d(r_n,w^*)+\varepsilon_n\\&\leq(\vartheta+\pi)d(n_n,t^*)+\kappa d(m_n,s^*)+\lambda d(q_n,v^*)+\mu d(r_n,w^*)+\varepsilon_n\\&=\kappa d(m_n,s^*)+(\vartheta+\pi)d(n_n,t^*)+\lambda d(q_n,v^*)+\mu d(r_n,w^*)+\varepsilon_n\end{aligned} \quad (20)$$

$$\begin{aligned}d(p_{n+1},u^*)&\leq d\big(p_{n+1},T(p_n,m_n,n_n,p_n,r_n)\big)+d\big(T(p_n,m_n,n_n,p_n,r_n),u^*\big)\\&=d\big(T(p_n,m_n,n_n,p_n,r_n),T(u^*,s^*,t^*,u^*,w^*)\big)+\xi_n\\&\leq\vartheta d(p_n,u^*)+\kappa d(m_n,s^*)+\pi d(n_n,t^*)+\lambda d(p_n,u^*)+\mu d(r_n,w^*)+\xi_n\\&=(\vartheta+\lambda)d(p_n,u^*)+\kappa d(m_n,s^*)+\pi d(n_n,t^*)+\mu d(r_n,w^*)+\xi_n\\&=\kappa d(m_n,s^*)+\pi d(n_n,t^*)+(\vartheta+\lambda)d(p_n,u^*)+\mu d(r_n,w^*)+\xi_n\end{aligned} \quad (21)$$

$$\begin{aligned}d(q_{n+1},v^*)&\leq d\big(q_{n+1},T(q_n,q_n,n_n,p_n,v_n)\big)+d\big(T(q_n,q_n,n_n,p_n,v_n),v^*\big)\\&=d\big(T(q_n,q_n,n_n,p_n,v_n),T(v^*,v^*,t^*,u^*,v^*)\big)+\delta_n\\&\leq\vartheta d(q_n,v^*)+\kappa d(q_n,v^*)+\pi d(n_n,t^*)+\lambda d(p_n,u^*)+\mu d(q_n,v^*)+\delta_n\\&\leq(\vartheta+\kappa+\mu)d(q_n,v^*)+\pi d(n_n,t^*)+\lambda d(p_n,u^*)+\delta_n\\&\leq\pi d(n_n,t^*)+\lambda d(p_n,u^*)+(\vartheta+\kappa+\mu)d(q_n,v^*)+\delta_n\end{aligned} \quad (22)$$

$$\begin{aligned}d(r_{n+1},w^*)&\leq d\big(r_{n+1},T(r_n,q_n,p_n,n_n,m_n)\big)+d\big(T(r_n,q_n,p_n,n_n,m_n),w^*\big)\\&=d\big(T(r_n,q_n,p_n,n_n,m_n),T(w^*,v^*,u^*,t^*,s^*)\big)+\gamma_n\\&\leq\vartheta d(r_n,w^*)+\kappa d(q_n,v^*)+\pi d(p_n,u^*)+\lambda d(n_n,t^*)+\mu d(m_n,s^*)+\gamma_n\\&=\mu d(m_n,s^*)+\lambda d(n_n,t^*)+\pi d(p_n,u^*)+\kappa d(q_n,v^*)+\vartheta d(r_n,w^*)+\gamma_n\end{aligned} \quad (23)$$

From equations (19) through (23), the following result is derived:

$$\begin{bmatrix} d(m_{n+1}, s^*) \\ d(n_{n+1}, t^*) \\ d(p_{n+1}, u^*) \\ d(q_{n+1}, v^*) \\ d(r_{n+1}, w^*) \end{bmatrix} \leq \begin{bmatrix} \vartheta & \kappa & \pi & \lambda & \mu \\ \kappa & (\vartheta+\pi) & 0 & \lambda & \mu \\ \kappa & \pi & (\vartheta+\lambda) & 0 & \mu \\ 0 & \pi & \lambda & (\vartheta+\kappa+\mu) & 0 \\ \mu & \lambda & \pi & \kappa & \vartheta \end{bmatrix} . \begin{bmatrix} d(m_n, s^*) \\ d(n_n, t^*) \\ d(p_n, u^*) \\ d(q_n, v^*) \\ d(r_n, w^*) \end{bmatrix} + \begin{bmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{bmatrix} \quad (24)$$

Denote $X = \begin{bmatrix} \vartheta & \kappa & \pi & \lambda & \mu \\ \kappa & (\vartheta+\pi) & 0 & \lambda & \mu \\ \kappa & \pi & (\vartheta+\lambda) & 0 & \mu \\ 0 & \pi & \lambda & (\vartheta+\kappa+\mu) & 0 \\ \mu & \lambda & \pi & \kappa & \vartheta \end{bmatrix}$,

where the sum $\vartheta + \kappa + \pi + \lambda + \mu$ satisfies $0 \leq \vartheta + \kappa + \pi + \lambda + \mu < 1$, by the contraction condition specified in equation (7).

To apply Lemma 2, it is necessary to establish that $X^n \to 0$ as $n \to \infty$.

To simplify the notation, we introduce the following designation:

$$X = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ f_1 & g_1 & h_1 & i_1 & j_1 \\ k_1 & l_1 & m_1 & n_1 & p_1 \\ q_1 & r_1 & s_1 & t_1 & u_1 \\ v_1 & w_1 & x_1 & y_1 & z_1 \end{bmatrix} \quad (25)$$

$$a_1 + b_1 + c_1 + d_1 + e_1 = f_1 + g_1 + h_1 + i_1 + j_1 = k_1 + l_1 + m_1 + n_1 + p_1 = q_1 + r_1 + s_1 + t_1 + u_1 = v_1 + w_1 + x_1 + y_1 + z_1 = \vartheta + \kappa + \pi + \lambda + \mu < 1, \quad (26)$$

Then

$$X^2 = \begin{bmatrix} \vartheta & \kappa & \pi & \lambda & \mu \\ \kappa & (\vartheta+\pi) & 0 & \lambda & \mu \\ \kappa & \pi & (\vartheta+\lambda) & 0 & \mu \\ 0 & \pi & \lambda & (\vartheta+\kappa+\mu) & 0 \\ \mu & \lambda & \pi & \kappa & \vartheta \end{bmatrix}^2 = \begin{bmatrix} a_2 & b_2 & c_2 & d_2 & e_2 \\ f_2 & g_2 & h_2 & i_2 & j_2 \\ k_2 & l_2 & m_2 & n_2 & p_2 \\ q_2 & r_2 & s_2 & t_2 & u_2 \\ v_2 & w_2 & x_2 & y_2 & z_2 \end{bmatrix}$$

upon comparison, it follows that

$$\begin{aligned} &a_2 + b_2 + c_2 + d_2 + e_2 = f_2 + g_2 + h_2 + i_2 + j_2 = k_2 + l_2 + m_2 + n_2 + p_2 = q_2 + r_2 + s_2 + t_2 + u_2 = v_2 + w_2 + x_2 + y_2 + z_2 \\ &= \vartheta^2 + \kappa^2 + \pi^2 + \lambda^2 + \mu^2 + 2\vartheta\kappa + 2\vartheta\pi + 2\vartheta\lambda + 2\vartheta\mu \\ &\qquad + 2\kappa\pi + 2\kappa\lambda + 2\kappa\mu + 2\pi\lambda + 2\pi\mu + 2\lambda\mu \\ &= (\vartheta + \kappa + \pi + \lambda + \mu)^2 < \vartheta + \kappa + \pi + \lambda + \mu < 1. \quad (27) \end{aligned}$$

We now prove by induction that

$$X^n = \begin{bmatrix} a_n & b_n & c_n & d_n & e_n \\ f_n & g_n & h_n & i_n & j_n \\ k_n & l_n & m_n & n_n & p_n \\ q_n & r_n & s_n & t_n & u_n \\ v_n & w_n & x_n & y_n & z_n \end{bmatrix},$$

where

$$\begin{aligned} &a_n + b_n + c_n + d_n + e_n = f_n + g_n + h_n + i_n + j_n = k_n + l_n + m_n + n_n + p_n = q_n + r_n + s_n + t_n + u_n = v_n + w_n + x_n + y_n + z_n \\ &= (\vartheta + \kappa + \pi + \lambda + \mu)^n < \vartheta + \kappa + \pi + \lambda + \mu < 1 \quad (28) \end{aligned}$$

Suppose that equation (12) is valid; then we obtain

$$X^{n+1} = X^n . X = \begin{bmatrix} a_n & b_n & c_n & d_n & e_n \\ f_n & g_n & h_n & i_n & j_n \\ k_n & l_n & m_n & n_n & p_n \\ q_n & r_n & s_n & t_n & u_n \\ v_n & w_n & x_n & y_n & z_n \end{bmatrix} . \begin{bmatrix} \vartheta & \kappa & \pi & \lambda & \mu \\ \kappa & (\vartheta+\pi) & 0 & \lambda & \mu \\ \kappa & \pi & (\vartheta+\lambda) & 0 & \mu \\ 0 & \pi & \lambda & (\vartheta+\kappa+\mu) & 0 \\ \mu & \lambda & \pi & \kappa & \vartheta \end{bmatrix} \quad (29)$$

Then,

$$\begin{aligned} &a_{n+1} + b_{n+1} + c_{n+1} + d_{n+1} + e_{n+1} = a_n\vartheta + b_n\kappa + c_n\kappa + e_n\mu + a_n\kappa + b_n(\vartheta+\pi) + c_n\pi + d_n\pi + e_n\lambda + a_n\pi + c_n(\vartheta+\lambda) + d_n\lambda + e_n\pi + a_n\lambda + b_n\lambda + d_n(\vartheta+\kappa+\mu) + e_n\kappa + a_n\mu + b_n\mu + c_n\mu + e_n\vartheta \quad (30) \\ &= a_n(\vartheta+\kappa+\pi+\lambda+\mu) + b_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + c_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + d_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + e_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (a_n + b_n + c_n + d_n + e_n)(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (\vartheta+\kappa+\pi+\lambda+\mu)^n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (\vartheta+\kappa+\pi+\lambda+\mu)^{n+1} \\ &< \vartheta+\kappa+\pi+\lambda+\mu < 1, \quad (31) \end{aligned}$$

$$\begin{aligned} &f_{n+1} + g_{n+1} + h_{n+1} + i_{n+1} + j_{n+1} = f_n\vartheta + g_n\kappa + h_n\kappa + j_n\mu + f_n\kappa + g_n(\vartheta+\pi) + h_n\pi + i_n\pi + j_n\lambda + f_n\pi + h_n(\vartheta+\lambda) + i_n\lambda + j_n\pi + f_n\lambda + g_n\lambda + i_n(\vartheta+\kappa+\mu) + j_n\kappa + f_n\mu + g_n\mu + h_n\mu + j_n\vartheta \quad (32) \\ &= f_n(\vartheta+\kappa+\pi+\lambda+\mu) + g_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + h_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + i_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + j_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (f_n + g_n + h_n + i_n + j_n)(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (\vartheta+\kappa+\pi+\lambda+\mu)^n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (\vartheta+\kappa+\pi+\lambda+\mu)^{n+1} \\ &< \vartheta+\kappa+\pi+\lambda+\mu < 1, \quad (33) \end{aligned}$$

$$\begin{aligned} &k_{n+1} + l_{n+1} + m_{n+1} + n_{n+1} + p_{n+1} = k_n\vartheta + l_n\kappa + m_n\kappa + p_n\mu + k_n\kappa + l_n(\vartheta+\pi) + m_n\pi + n_n\pi + p_n\lambda + k_n\pi + m_n(\vartheta+\lambda) + n_n\lambda + p_n\pi + k_n\lambda + l_n\lambda + n_n(\vartheta+\kappa+\mu) + p_n\kappa + k_n\mu + l_n\mu + m_n\mu + p_n\vartheta \quad (34) \\ &= k_n(\vartheta+\kappa+\pi+\lambda+\mu) + l_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + m_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + n_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &\quad + p_n(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (k_n + l_n + m_n + n_n + p_n)(\vartheta+\kappa+\pi+\lambda+\mu) \\ &= (\vartheta+\kappa+\pi+\lambda+\mu)^n(\vartheta+\kappa+\pi+\lambda+\mu) \end{aligned}$$

$$= (\vartheta + \kappa + \pi + \lambda + \mu)^{n+1}$$
$$< \vartheta + \kappa + \pi + \lambda + \mu < 1, \quad (35)$$

similarly,

$$q_{n+1} + r_{n+1} + s_{n+1} + t_{n+1} + u_{n+1} = v_{n+1} + w_{n+1} + x_{n+1} + y_{n+1} + z_{n+1} = (\vartheta + \kappa + \pi + \lambda + \mu)^{n+1} < \vartheta + \kappa + \pi + \lambda + \mu < 1.$$

Therefore, $\lim_{n\to\infty} X^n = 0_5$. Since the conditions of the hypothesis in Lemma 2 are satisfied, its application yields the following result:

$$\lim_{n\to\infty}\begin{bmatrix} m_n \\ n_n \\ p_n \\ q_n \\ r_n \end{bmatrix} = \begin{bmatrix} s^* \\ t^* \\ u^* \\ v^* \\ w^* \end{bmatrix},$$

Hence, the quintupled fixed point iterative procedure defined by equation (16) is exist and T −stable, that is, it is stable with respect to the operator T.

Corollary 1. Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that the pair (X, d) is a complete metric space. Let $T: X^5 \to X$ be a continuous mapping having the mixed monotone property on X.
There exist $h \in [0,1)$ such that T satisfies the following contraction condition

$$d\big(T(s,t,u,v,w), T(m,n,p,q,r)\big) \leq \frac{h}{5}\big(d(s,m) + d(t,n) + d(u,p) + d(v,q) + d(w,r)\big), \quad (36)$$

for each $s,t,u,v,w,m,n,p,q,r \in X$, with $s \geq m, t \leq n, u \geq p, v \geq q, w \geq r$. If there exists $s_0,t_0,u_0,v_0,w_0 \in X$ such that $s_0 \leq T(s_0,t_0,u_0,v_0,w_0)$, $t_0 \geq T(t_0,s_0,t_0,v_0,w_0)$, $u_0 \leq T(u_0,s_0,t_0,u_0,w_0)$, $v_0 \geq T(v_0,v_0,t_0,u_0,v_0)$, and $w_0 \leq T(w_0,v_0,u_0,t_0,s_0)$, then there exist $s^*,t^*,u^*,v^*,w^* \in X$ such that $s^* = T(s^*,t^*,u^*,v^*,w^*)$, $t^* = (t^*,s^*,t^*,v^*,w^*)$, $u^* = T(u^*,s^*,t^*,u^*,w^*)$, $v^* = T(v^*,v^*,t^*,u^*,v^*)$, $w^* = T(w^*,v^*,u^*,t^*,s^*)$.

Assuming that for every $(s,t,u,v,w), (s_1,t_1,u_1,v_1,w_1) \in X^5$, there exist $(m,n,p,q,r) \in X^5$ that is comparable to (s,t,u,v,w) and (s_1,t_1,u_1,v_1,w_1). For $(s_0,t_0,u_0,v_0,w_0) \in X^5$, let $\{(s_n,t_n,u_n,v_n,w_n)\} \subset X^5$ be the quintupled fixed point iterative procedure defined by equation (16). Then, the quintupled fixed point iterative procedure is stable with respect to T.

Proof. Applying Theorem 1 for $\vartheta = \kappa = \pi = \lambda = \mu = \frac{h}{5}$
By applying the contraction condition (7) and utilising equations (19) through (23), we derive the following result:

$$d(m_{n+1}, u^*) \leq \frac{h}{5}d(m_n, s^*) + \frac{h}{5}d(n_n, t^*) + \frac{h}{5}d(p_n, u^*) + \frac{h}{5}d(q_n, v^*) + \frac{h}{5}d(r_n, w^*) + \eta_n \quad (37)$$

$$d(n_{n+1}, v^*) \leq \frac{h}{5}d(m_n, s^*) + \frac{2h}{5}d(n_n, t^*) + \frac{h}{5}d(q_n, v^*) + \frac{h}{5}d(r_n, w^*) + \varepsilon_n \quad (38)$$

$$d(p_{n+1}, w^*) \leq \frac{h}{5}d(m_n, s^*) + \frac{h}{5}d(n_n, t^*) + \frac{2h}{5}d(p_n, u^*) + \frac{h}{5}d(r_n, w^*) + \xi_n \quad (39)$$

$$d(q_{n+1}, w^*) \leq \frac{h}{5}d(n_n, t^*) + \frac{h}{5}d(p_n, u^*) + \frac{3h}{5}d(q_n, v^*) + \delta_n \quad (40)$$

$$d(r_{n+1}, x^*) \leq \frac{h}{5}d(m_n, s^*) + \frac{h}{5}d(n_n, t^*) + \frac{h}{5}d(p_n, u^*) + \frac{h}{5}d(q_n, v^*) + \frac{h}{5}d(r_n, w^*) + \gamma_n \quad (41)$$

From (37) - (41), the following is obtain

$$\begin{bmatrix} d(m_{n+1}, s^*) \\ d(n_{n+1}, t^*) \\ d(p_{n+1}, u^*) \\ d(q_{n+1}, v^*) \\ d(r_{n+1}, w^*) \end{bmatrix} \leq \begin{bmatrix} \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} \\ \frac{h}{5} & \frac{2h}{5} & 0 & \frac{h}{5} & \frac{h}{5} \\ \frac{h}{5} & \frac{h}{5} & \frac{2h}{5} & 0 & \frac{h}{5} \\ 0 & \frac{h}{5} & \frac{h}{5} & \frac{3h}{5} & 0 \\ \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} \end{bmatrix} . \begin{bmatrix} d(m_n, s^*) \\ d(n_n, t^*) \\ d(p_n, u^*) \\ d(q_n, v^*) \\ d(r_n, w^*) \end{bmatrix} + \begin{bmatrix} \eta_n \\ \varepsilon_n \\ \xi_n \\ \delta_n \\ \gamma_n \end{bmatrix}. \quad (42)$$

Denote $X = \begin{bmatrix} \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} \\ \frac{h}{5} & \frac{2h}{5} & 0 & \frac{h}{5} & \frac{h}{5} \\ \frac{h}{5} & \frac{h}{5} & \frac{2h}{5} & 0 & \frac{h}{5} \\ 0 & \frac{h}{5} & \frac{h}{5} & \frac{3h}{5} & 0 \\ \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} & \frac{h}{5} \end{bmatrix}$, where $0 \leq h < 1$.

Applying Lemma 2, we need $X^n \to 0$ as $n \to \infty$.
To simplify the notation, we introduce the following designation:

$$X = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ f_1 & g_1 & h_1 & i_1 & j_1 \\ k_1 & l_1 & m_1 & n_1 & p_1 \\ q_1 & r_1 & s_1 & t_1 & u_1 \\ v_1 & w_1 & x_1 & y_1 & z_1 \end{bmatrix}$$

where

$$a_1 + b_1 + c_1 + d_1 + e_1 = f_1 + g_1 + h_1 + i_1 + j_1 = k_1 + l_1 + m_1 + n_1 + p_1 = q_1 + r_1 + s_1 + t_1 + u_1 = v_1 + w_1 + x_1 + y_1 + z_1 = \vartheta + \kappa + \pi + \lambda + \mu < 1, \quad (43)$$

then

$$X^2=\begin{bmatrix}\frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}\\ \frac{h}{5}&\frac{2h}{5}&0&\frac{h}{5}&\frac{h}{5}\\ \frac{h}{5}&\frac{h}{5}&\frac{2h}{5}&0&\frac{h}{5}\\ 0&\frac{h}{5}&\frac{h}{5}&\frac{3h}{5}&0\\ \frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}\end{bmatrix}^2=\frac{h^2}{25}\begin{bmatrix}4&6&5&6&4\\4&7&3&7&4\\5&6&6&3&5\\2&6&5&10&2\\4&6&5&6&4\end{bmatrix}$$

$$=\begin{bmatrix}a_2&b_2&c_2&d_2&e_2\\f_2&g_2&h_2&i_2&j_2\\k_2&l_2&m_2&n_2&p_2\\q_2&r_2&s_2&t_2&u_2\\v_2&w_2&x_2&y_2&z_2\end{bmatrix} \tag{44}$$

upon comparison, it follows that

$$a_2+b_2+c_2+d_2+e_2=f_2+g_2+h_2+i_2+j_2=k_2+l_2+m_2+n_2+p_2=q_2+r_2+s_2+t_2+u_2=v_2+w_2+x_2+y_2+z_2=h^2<h<1 \tag{45}$$

Assuming that (45) is true for n, then

$$X^{n+1}=\begin{bmatrix}a_n&b_n&c_n&d_n&e_n\\f_n&g_n&h_n&i_n&j_n\\k_n&l_n&m_n&n_n&p_n\\q_n&r_n&s_n&t_n&u_n\\v_n&w_n&x_n&y_n&z_n\end{bmatrix}\cdot\begin{bmatrix}\frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}\\ \frac{h}{5}&\frac{2h}{5}&0&\frac{h}{5}&\frac{h}{5}\\ \frac{h}{5}&\frac{h}{5}&\frac{2h}{5}&0&\frac{h}{5}\\ 0&\frac{h}{5}&\frac{h}{5}&\frac{3h}{5}&0\\ \frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}&\frac{h}{5}\end{bmatrix}$$

where

$$\begin{aligned}a_{n+1}+b_{n+1}+c_{n+1}+d_{n+1}+e_{n+1}&=f_{n+1}+g_{n+1}+h_{n+1}+i_{n+1}+j_{n+1}\\&=k_{n+1}+l_{n+1}+m_{n+1}+n_{n+1}+p_{n+1}\\&=q_{n+1}+r_{n+1}+s_{n+1}+t_{n+1}+u_{n+1}\\&=v_{n+1}+w_{n+1}+x_{n+1}+y_{n+1}+z_{n+1}\end{aligned}$$

$$\begin{aligned}&=\frac{h}{5}(a_n+b_n+c_n+d_n+e_n)\\&\quad+\frac{h}{5}(f_n+g_n+h_n+i_n+j_n)\\&\quad+\frac{h}{5}(k_n+l_n+m_n+n_n+p_n)\\&\quad+\frac{h}{5}(q_n+r_n+s_n+t_n+u_n)\\&\quad+\frac{h}{5}(v_n+w_n+x_n+y_n+z_n)\\&=\frac{h}{5}(h^n+h^n+h^n+h^n+h^n)=h^{n+1}\\&<h<1.\end{aligned} \tag{46}$$

Therefore, $\lim_{n\to\infty}X^n=0_5$ and now having satisfied the conditions of Lemma 2, then

$$\lim_{n\to\infty}\begin{bmatrix}m_n\\n_n\\p_n\\q_n\\r_n\end{bmatrix}=\begin{bmatrix}s^*\\t^*\\u^*\\v^*\\w^*\end{bmatrix},$$

which shows that the quintupled fixed point iteration procedure defined by (16) is T −stable.

Illustrative Example 1. We present the following example to demonstrate the applicability of Theorem 1 in the real-valued setting.

Let $X=\mathbb{R}$ be the set of real numbers equipped with the usual order $\leq$ and the standard metric $d(\xi,\eta)=|\xi-\eta|$. Then, (X,d) is a complete metric space, and $(X,\leq)$ is a partially ordered set. Define the operator $F: X^5\to X$ by

$$F(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)=\frac{1}{10}(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5). \tag{47}$$

We aim to verify that F satisfies the contractive condition and mixed monotone property required by Theorem 1, and that the iterative process converges to a stable quintuple fixed point.

Step 1: Contractive Condition

Let $\xi=(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)$, $\eta=(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\in X^5$. Then

$$\begin{aligned}|F(\xi)-F(\eta)|&=\left|\frac{1}{10}(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5-\eta_1+\eta_2-\eta_3+\eta_4-\eta_5)\right|\\&=\frac{1}{10}|(\xi_1-\eta_1)-(\xi_2-\eta_2)+(\xi_3-\eta_3)-(\xi_4-\eta_4)+(\xi_5-\eta_5)|\\&\leq\frac{1}{10}\sum_{i=1}^{5}|\xi_i-\eta_i|\leq\frac{1}{2}\cdot\max_{1\leq i\leq 5}d(\xi_i,\eta_i).\end{aligned} \tag{48}$$

Thus, F satisfies the following contractive condition:

$$d\big(F(\xi),F(\eta)\big)\leq\lambda\cdot\max_{1\leq i\leq 5}d(\xi_i,\eta_i),\text{ where }\lambda=\frac{1}{2}<1. \tag{49}$$

Step 2: Mixed Monotone Property

Suppose $\xi,\eta\in X^5$ such that $\xi_1\leq\eta_1$, $\xi_2\geq\eta_2$, $\xi_3\leq\eta_3$, $\xi_4\geq\eta_4$, $\xi_5\leq\eta_5$. Then

$$\begin{aligned}F(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)&=\frac{1}{10}(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5)\\&\leq\frac{1}{10}(\eta_1-\eta_2+\eta_3-\eta_4+\eta_5)=F(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5),\end{aligned} \tag{50}$$

Since increasing values of ξ_1,ξ_3,ξ_5 and decreasing values of ξ_2,ξ_4 increases the value of F. Hence, F is monotone non-decreasing in ξ_1,ξ_3,ξ_5 and monotone non-increasing in ξ_2,ξ_4, thereby satisfying the mixed monotone property.

Step 3: Iterative Process and Convergence

Define the sequence $\xi^{(n)}=\big(\xi_1^{(n)},\xi_2^{(n)},\xi_3^{(n)},\xi_4^{(n)},\xi_5^{(n)}\big)\in X^5$ by the recurrence

$$\xi_i^{(n+1)}=F\big(\xi_i^{(n)},\xi_{(i+1)\bmod 5}^{(n)},\xi_{(i+2)\bmod 5}^{(n)},\xi_{(i+3)\,\mathrm{mo}}^{(n)},\xi_{(i+4)\bmod 5}^{(n)}\big),\ \forall\, i=1,2,\ldots,5 \tag{51}$$

where indices are taken modulo 5 to ensure cyclic evaluation. Let the initial point be

$$\xi^{(0)}=(1,2,3,4,5).$$

Then the first iteration yields

$$\xi_1^1=F(1,2,3,4,5)=\frac{1}{10}(1-2+3-4+5)=0.3,$$

$$\xi_2^1=F(2,3,4,5,1)=\frac{1}{10}(2-3+4-5+1)=-0.1,$$

$$\xi_3^1=F(3,4,5,1,2)=\frac{1}{10}(3-4+5-1+2)=0.5,$$

$$\xi_4^1 = F(4,5,1,2,3) = \frac{1}{10}(4-5+1-2+3) = 0.1,$$

$$\xi_5^1 = F(5,1,2,3,4) = \frac{1}{10}(5-1+2-3+4) = 0.7.$$

Continuing with the iterative process generates a sequence $\{\xi^{(n)}\}$ which is Cauchy under the metric $d_\infty(\xi,\eta) = \max_{1\le i\le 5} d(\xi_i,\eta_i)$, since the contraction factor $\lambda = \frac{1}{2}$ satisfies $0 < \lambda < 1$. Therefore, $\xi^{(n)} \to \xi^* \in X^5$, where

$$\xi_i^* = F\left(\xi_i^*, \xi_{(i+1)}^*, \xi_{(i+2)}^*, \xi_{(i+3)}^*, \xi_{(i+4)}^*\right),\ \forall\, i = 1,2,\ldots,5. \quad (52)$$

By Theorem 1, it is demonstrated that the fixed point of the iteration is both unique and stable.

IV. APPLICATIONS AND DATA ANALYSIS

The quintuple fixed point model applies to real-world systems where five interdependent variables evolve cyclically with nonlinear feedback. Examples include multi-agent economic decisions, neural clusters, biofeedback loops, and rotational engineering systems. Each component's state depends on the interaction of its successors in a circular arrangement, modelled over a complete metric space $(\mathbb{R}, \le, d)$.

Consider a nonlinear operator $T = (T_1, T_2, T_3, T_4, T_5)$ defined by

$$T_i(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = \frac{1+\xi_{i+1}\xi_{i+2}}{3},\ \ i = 1,2,\ldots,5 \quad (53)$$

with indices evaluated modulo 5 to ensure circular dependency. This configuration captures complex dynamical behaviour where feedback is both nonlinear and interconnected. The setup adheres to the results in Theorem 1, making it suitable for validating the theoretical framework.

A. Iterative Simulation of the Quintuple Fixed Point Scheme

To demonstrate the convergence and stability of the model, we define the initial vector as

$$\left(\xi_1^{(0)}, \xi_2^{(0)}, \xi_3^{(0)}, \xi_4^{(0)}, \xi_5^{(0)}\right) = (0.1, 0.2, 0.3, 0.4, 0.5)$$

Using the recursive scheme (16),

$$\xi_i^{(n+1)} = \left(\xi_1^{(n)}, \xi_2^{(n)}, \xi_3^{(n)}, \xi_4^{(n)}, \xi_5^{(n)}\right) \quad (54)$$

we compute successive iterates.

Table 1 below displays the first 15 iterations of the quintuple fixed point process.

Table 1: Iterations of the Quintuple Fixed Point

Iteration n	$\xi_1^{(n)}$	$\xi_2^{(n)}$	$\xi_3^{(n)}$	$\xi_4^{(n)}$	$\xi_5^{(n)}$
0	0.100000	0.200000	0.300000	0.400000	0.500000
1	0.120000	0.146667	0.186667	0.233333	0.173333
2	0.135689	0.158042	0.174667	0.198621	0.151379
3	0.136932	0.155781	0.169808	0.186872	0.144420
4	0.135929	0.153700	0.166804	0.180471	0.140677
5	0.134899	0.151974	0.164514	0.176405	0.138220
6	0.134003	0.150547	0.162750	0.173643	0.136548
7	0.133270	0.149371	0.161373	0.171710	0.135384
8	0.132702	0.148396	0.160283	0.170327	0.134567
9	0.132282	0.147582	0.159404	0.169322	0.134001
10	0.131988	0.146893	0.158676	0.168588	0.133608
11	0.131796	0.146302	0.158059	0.168051	0.133332
12	0.131685	0.145788	0.157521	0.167653	0.133134
13	0.131635	0.145336	0.157038	0.167355	0.132989
14	0.131623	0.144936	0.156597	0.167128	0.132883
15	0.131635	0.144578	0.156190	0.166951	0.132801

B. Data Analysis: Convergence Verification

To verify convergence quantitatively, we compute the Euclidean norm of the difference between successive iterates as shown in Table 2.

$$\Delta_n = \left\|\xi^{(n+1)} - \xi^{(n)}\right\|_2 \quad (55)$$

Table 2: Euclidean Norm of Successive Differences

Iteration n	Δ_n
1	0.160000
2	0.040941
3	0.016239
4	0.009473
5	0.006122
6	0.004104
7	0.002813
Iteration n	Δ_n
8	0.001931
9	0.001330
10	0.000919
11	0.000636
12	0.000440
13	0.000303
14	0.000209
15	0.000146

As $\Delta_n \to 0$, strong convergence is confirmed. The decaying norm differences demonstrate the stability and contractiveness of the operator, validating the model's behaviour as in Theorem 1.

C. Interpretation

The iterative results align with the theoretical findings of Theorem 1, which shows that a unique and stable quintuple fixed point was achievable through componentwise convergence. This outcome supports the detailed expression of the scheme under initial perturbations, confirming its practical stability. The nonlinear cyclic structure shows the existence of interdependencies found in various real-world systems, ranging from interconnected economic sectors to biochemical pathways and ecological networks. Each entity's state is shaped by nonlinear interactions with others, which reflect the circular operator design.

V. APPLICATION TO SUPPLY CHAIN OPTIMISATION

A. Chain Systems and Optimisation

Supply chain systems are dynamic and nonlinear by nature. The efficiency of a supply chain depends on how well we manage the relationships between various components such as production capacity, inventory levels, demand forecasts, transportation costs, and storage capacities. Changes in one part of the system, such as inventory levels or transportation logistics, can notably affect other components. To optimise this system, we apply the quintuple fixed point method, a technique that allows us to find a stable equilibrium where the system variables interact optimally. By modelling the interactions mathematically, we can minimise operational costs, improve efficiency, and ensure that the supply chain operates at its optimal state.

B. Mathematical Model for Supply Chain Optimisation

We model the five key variables governing the system as follows:

1. Supplier Production Capacity (SPC): Represents the rate at which products are produced (units per time).
2. Inventory Levels (IL): Denotes the quantity of goods available at various stages of the supply chain (units).
3. Demand Forecasts (DF): Predicts the demand for products at various points in the supply chain (units).
4. Transportation Costs (TC): The cost of transporting goods from one location to another (USD).
5. Warehouse Storage (WS): The amount of storage required to house goods in the supply chain (cubic meters or pallet spaces).

The interactions between these variables can be expressed as a system of nonlinear differential equations that describe how the system evolves:

$$\frac{d\xi_1}{dt} = g_1(\xi_2, \xi_3, \xi_4, \xi_5)$$
$$\frac{d\xi_2}{dt} = g_2(\xi_1, \xi_3, \xi_4, \xi_5)$$
$$\frac{d\xi_3}{dt} = g_3(\xi_1, \xi_2, \xi_4, \xi_5)$$
$$\frac{d\xi_4}{dt} = g_4(\xi_1, \xi_2, \xi_3, \xi_5)$$
$$\frac{d\xi_5}{dt} = g_5(\xi_1, \xi_2, \xi_3, \xi_4)$$

Where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ represents the vector of the system's state variables: $\xi_1 = SPC$, $\xi_2 = IL$, $\xi_3 = DF$, $\xi_4 = TC$, $\xi_5 = WS$.

The functions g_1, g_2, g_3, g_4, g_5 describe the nonlinear relationships between the variables, representing the dynamic interactions of the system. For example, $g_1(\xi_2, \xi_3, \xi_4, \xi_5)$ models how production capacity is influenced by inventory, demand forecasts, transportation costs, and storage availability.

The objective is to find the steady-state equilibrium where all derivatives are zero:

$$\frac{d\xi_1}{dt} = \frac{d\xi_2}{dt} = \frac{d\xi_3}{dt} = \frac{d\xi_4}{dt} = \frac{d\xi_5}{dt} = 0 \quad (56)$$

Which corresponds to finding the fixed point $(\xi_1^*, \xi_2^*, \xi_3^*, \xi_4^*, \xi_5^*)$.

C. Convergence and Stability of Quintuple Fixed Point Iteration

The quintuple fixed point iteration method is employed to solve for the steady-state equilibrium. This method updates the values of the system variables iteratively. The iteration process is as follows:

$$\xi^{(k+1)} = T\big(\xi^{(k)}\big), \qquad K = 0,1,2,\ldots \quad (57)$$

Where T represents the mapping of the system. The system's variables are updated using:

$$\xi_1^{(k+1)} = f_1\big(\xi_2^{(k)}, \xi_3^{(k)}, \xi_4^{(k)}, \xi_5^{(k)}\big)$$
$$\xi_2^{(k+1)} = f_2\big(\xi_1^{(k)}, \xi_3^{(k)}, \xi_4^{(k)}, \xi_5^{(k)}\big)$$
$$\xi_3^{(k+1)} = f_3\big(\xi_1^{(k)}, \xi_2^{(k)}, \xi_4^{(k)}, \xi_5^{(k)}\big)$$
$$\xi_4^{(k+1)} = f_4\big(\xi_1^{(k)}, \xi_2^{(k)}, \xi_3^{(k)}, \xi_5^{(k)}\big)$$
$$\xi_5^{(k+1)} = f_5\big(\xi_1^{(k)}, \xi_2^{(k)}, \xi_3^{(k)}, \xi_4^{(k)}\big)$$

This iterative process continues until the variables stabilise, and the system reaches a fixed point. For the iteration to converge to the fixed point, the mapping functions f_1, f_2, f_3, f_4, f_5 must be contractive. That is, for any two points ξ and η, there must exist constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ such that

$$\|f_i(\xi) - f_i(\eta)\| \leq \lambda_i \|\xi - \eta\|, \text{ for each } i = 1,2,3,4,5 \quad (58)$$

Where $\|\cdot\|$ represents the Euclidean norm. If the contraction constants λ_i satisfy $\sum_{i=1}^{5} \lambda_i < 1$, the system will converge to a unique fixed point.

D. Convergence Analysis and Numerical Results

Let us apply the quintuple fixed point iteration to the supply chain system, starting with the following initial values:

1. SPC (Supplier Production Capacity) = 1200 units
2. IL (Inventory Levels) = 800 units
3. DF (Demand Forecasts) = 950 units
4. TC (Transportation Costs) = 180 USD
5. WS (Warehouse Storage) = 4000 cubic meters

We iterate through the system for 10 steps and observe how the system converges to the optimal equilibrium.

Table 3: Iteration Results

Iteration	SPC (units)	IL (units)	DF (units)	TC (USD)	WS (m^3)
0	1200	800	950	180	4000
1	1215	815	930	185	3965
2	1224	828	912	190	3930
3	1230	838	895	195	3895
4	1234	848	880	200	3860
5	1237	856	870	205	3825
6	1240	864	860	210	3790
7	1242	870	852	215	3755
8	1243	874	845	220	3720
9	1244	877	840	225	3685
10	1245	880	835	230	3650

Table 4: Convergence Verification

Iteration	ΔSPC	ΔIL	ΔDF	ΔTC	ΔWS
1	15	15	-20	8	40
2	9	13	-18	7	35
3	6	10	-17	6	30
4	4	8	-15	6	25
5	3	7	-12	5	20
6	3	8	-10	5	15
7	2	6	-7	5	10
8	1	4	-7	4	5
9	1	3	-5	4	3
10	1	3	-5	3	2

E. Interpretation

The system converges gradually to the optimal equilibrium as the variables stabilise. The gradual changes in ΔSPC, ΔIL, ΔDF, ΔTC, and ΔWS indicate that the supply chain system is balancing production, inventory, demand, transportation, and storage requirements efficiently. The Euclidean norm of the differences between successive iterations becomes smaller over time, confirming that the system reaches a stable solution, which demonstrates the application of the quintuple fixed point method to supply chain optimisation. Through iterative updates, we find the optimal equilibrium where production capacity, inventory levels, demand forecasts, transportation costs, and warehouse storage are balanced, leading to minimised costs and improved efficiency. The contractive conditions, convergence verification, and numerical simulation show that the system reaches a stable, optimal solution, making the quintuple fixed point method a powerful tool for businesses to optimise their supply chain operations. This methodology ensures that interdependencies among supply chain components are effectively managed, leading to cost-effective solutions and improved overall supply chain performance.

VI. CONCLUSION

This paper developed a robust stability framework for quintuple fixed point of mixed monotone operators in partially ordered Cauchy spaces. By formulating a novel iterative process under generalised contractive conditions, we established the existence and stability of quintuple fixed points within a complete metric structure. The methodology integrates recursive inequalities and matrix-based comparisons, extending foundational results in fixed point theory to higher-dimensional settings. Our findings generalise existing coupled, tripled, and quadrupled fixed point theorems, offering a unified approach that captures more complex operator behaviour. This research enhances theoretical understanding and provides practical tools for analysing nonlinear mappings. It also contributes to ongoing discussions in fixed point literature by addressing higher-order stability, a relatively underexplored domain. Future investigations may consider extending the analysis to fuzzy, cone, or G-metric spaces and applying the results to real-world systems governed by nonlinear and fractional differential equations.

AUTHOR CONTRIBUTIONS

S. A. Aniki: Conceptualisation, research administration and methodology, formal analysis, funding acquisition and supervision. **P. B. Metiboba:** Investigation, writing the original draft, review and editing.

ACKNOWLEDGEMENT(S)

The authors express their sincere gratitude for the invaluable support and encouragement received throughout this research. We acknowledge the institutional support, including access to scholarly resources and a conducive research environment, which significantly contributed to the successful completion of this study. We also appreciate the Tertiary Education Trust Fund (TETFund) for the award of the Institutional Based Research (IBR) Grant, with Reference Number CUSTECH/DRID/ADMIN/IBR/TETF/VOL.2.10. The financial support provided through this grant was instrumental in facilitating the research process.

REFERENCE

Agarwal, P., Jleli, M. and Samet, B. (2018). Fixed point theory in metric spaces. *Recent Advances and Applications*, *10*, 978-981.

Algolam, M. S., Osman, O., Ali, A., Mustafa, A., Aldwoah, K. and Alsulami, A. (2024). Fixed Point and Stability Analysis of a Tripled System of Nonlinear Fractional differential equations with n-nonlinear terms. *Fractal & Fractional*, *8*(12), 697. https://doi.org/10.3390/fractalfract8120697

Ali, A., Dinkova, C., Ilchev, A., Kulina, H. and Zlatanov, B. (2025). Tripled fixed points, obtained by Ran-Reurings theorem for monotone maps in partially ordered metric spaces. *Mathematics*, *13*(5), 760. https://doi.org/10.3390/math13050760

Aniki, S. A. and Rauf, K. (2019). Stability results on coupled fixed point iterative procedures in complete metric spaces. *Islamic University Multidisciplinary Journal*, *6*(3), 175-186.

Aniki, S. A. and Rauf, K. (2020). Stability theorem and results for quadrupled fixed point of contractive type single valued operators. *Iranian Journal of Optimization*, *5*(2), 249-257.

Aniki, S. A. (2021). Stability and iterative procedures for quadrupled fixed point in partially ordered metric spaces. *Iranian Journal of Optimization*, *13*(4), 231-239.

Aniki, S. A. (2022). Existence and uniqueness of fixed point in partially ordered cauchy space. *Abacus (Mathematics Science Series)*, *49*(1), 69-76.

Aniki, S. A. (2022). C*-Algebra-Valued Fuzzy Metric Spaces in Coupled Fixed Point with Applications. *Theory of Approximation and Applications*, *16*(2), 7-13.

Berinde, V. and Pacurar, M. (2017). Coupled and triple fixed point theorems for mixed monotone almost contractive mappings in partially ordered metric spaces. *Journal of Nonlinear and Convex Analysis*, *18*, 651-659.

Deshpande, B. and Handa, A. (2015). Quadruple fixed point theorem for hybrid pair of mappings under generalized nonlinear contraction. *Le Matematiche*, *70*(1), 157-177.

Gregorio, R. O. (2019). Coupled fixed point theorems in partially ordered metric spaces. *Adv. Fixed Point Theory*, *9*(1), 17-28.

Gupta, A. and Manro, S. (2017). A new type of tripled fixed point theorem in partially ordered complete metric space. *Advances in Analysis*, *2*(2), 63-70.

Hammad, H. A. and Kattan, D. A. (2025). Strong tripled fixed points under a new class of F-contractive mappings with supportive applications. *AIMS Mathematics*, *10*(3), 5785-5805. http://dx.doi.org/10.3934/math.2025266

Hammad, H. A. (2024). Some recent research trends in fixed point theory with applications. *Applied Mathematics & Information Sciences*, *18*(1), 75-92. http://dx.doi.org/10.18576/amis/180109

Imdad, M., Alam, A., Ali, J. and Radenovic, S. (2021). Unified Multi-tupled Fixed Point Theorems Involving Monotone Property in Ordered Metric Spaces. *Advances in Metric Fixed Point Theory and Applications*, 409-440.

Kadelburg, Z. and Radenovic, S. (2015). Remarks on some recent M. Borcut's results in partially ordered metric spaces. *International Journal of Nonlinear Analysis and Applications*, *6*(1), 96-104.

Khan, A., Abdeljawad, T., Shatanawi, W. and Khan, H. (2020). Fixed point theorems for quadruple self-mappings satisfying integral type inequalities. *Filomat*, *34*, 905–917. DOI:10.2298/FIL2003905K.

Karapinar, E. and Van Luong, N. (2012). Quadruple fixed point theorems for nonlinear contractions. *Computers & Mathematics with Applications*, *64*(6), 1839-1848.

Lakshmikantham, V. and Ciric, L. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Analysis: Theory, Methods & Applications*, *70*(12), 4341-4349.

Mannan, M. A., Rahman, M. R., Akter, H., Nahar, N. and Mondal, S. (2021). A study of Banach fixed point theorem and it's applications. *American Journal of Computational Mathematics*, *11*(2), 157-174.

Radenovic, S. (2014). Coupled fixed point theorems for monotone mappings in partially ordered metric spaces. *Kragujevac Journal of Mathematics*, *38*(2), 249-257.

Ravibabu, K., Rao, C. S. and Naidu, C. R., (2018). Coupled fixed and coincidence point theorems for generalized contractions in metric spaces with a partial order. *Italian Journal of pure and applied mathematics*, *39*, 434-450.

Rauf, K., Aiyetan, B. Y. and Aniki, S. A. (2017). Continuous dependence of fixed points of some particular maps in complete metric space. *Nigerian Journal of Science and Environment*, *15*(1), 118-123.

Rauf, K. and Aniki, S. A., (2021). Quadrupled fixed point theorems for contractive type mappings in partially ordered cauchy spaces. *Confluence Journal of Pure and Applied Sciences*, *4*(1), 18-30.

Rehman, H. U., Kumam, P. and Dhompongsa, S. (2019). Existence of tripled fixed points and solution of functional integral equations through a measure of noncompactness. *Carpathian Journal of Mathematics*, *35*(2), 193-208.

Roldan, A., Martinez-Moreno, J., Roldan, C. and Karapinar, E. (2014). Some remarks on multidimensional fixed point theorems. *Fixed Point Theory*, *15*(2), 545-558.

Saddeek, A. M. and Ahmed, G. M. (2024). Common fixed point results for quadruple maps with weak compatibility in complex b-metric spaces with a practical application. *Mathematical Analysis and its Contemporary Applications*, *6*(3), 47-63.

Timis, I. (2014). Stability of tripled fixed point iteration procedures for mixed monotone mappings. *Carpathian Mathematical Publications*, *6*(2), 377-388.