

An Improved Numerical Analysis of Couette-Poiseuille Flow Using Hybrid Shooting-Differential Transform Method



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ABSTRACT: Couette-Poiseuille flow is a fundamental concept in fluid mechanics that involves the combined effects of shear and pressure-driven forces in a channel that has a wide range of engineering applications such as lubrication, microfluidics, and heat exchangers. Appropriate methodological mechanism is important in the analysis of the effects and behaviour of such systems. Conventional numerical methods frequently encounter stability and convergence issues, particularly when dealing with nonlinear or stiff-flow problems. This study presents the application of the Modified Shooting Method (MSM) to solve and analyse the differential equation resulting from the Couette-Poiseuille Flow. The DTM reduces differential equations to a set of algebraic equations, providing improved efficiency and accuracy. On the other hand, the MSM boosts stability and convergence compared to conventional shooting procedures. The hybrid shooting-differential transform method combines the two techniques and leverages the strengths of both, creating a strong solution framework for the nonlinear differential equations that drive Couette-Poiseuille flow. The equations were first transformed into algebraic equations using the appropriate transforms. Finally, the shooting numerical scheme was deployed to solve the transformed equations. The results of our study show that the hybrid method achieved a percentage error of < 10-8% and an improved computing efficiency of >70%.

KEYWORDS: Couette-Poiseuille flow, Differential Transform Method, Modified Shooting Method, Fluid Mechanics, Nonlinear Differential Equations

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I. INTRODUCTION

Fluid dynamics, specifically the analysis of Couette-Poiseuille flow, is essential for understanding how fluids behave in different engineering contexts. Couette-Poiseuille flow is a well-known phenomenon in fluid mechanics that combines both shear-driven (Couette) and pressure-driven (Poiseuille) flow in a channel. It has several applications in engineering, such as lubrication theory, polymer processing, and microfluidic devices (White, 2006). To comprehend the velocity profiles and pressure distribution in these flows, it is necessary to solve the Navier-Stokes equations. The utilisation of this hybrid flow model is essential for the purpose of planning and optimising systems that involve the simultaneous presence of both shear and pressure gradients. These equations are complicated and typically require advanced analytical and computational approaches, as stated by Batchelor (2000) and White (2006).

Analytical approaches yield precise solutions, but their applicability is often limited to uncomplicated geometries and boundary conditions, rendering them less feasible for complex situations. In contrast, numerical techniques such as the finite difference method, finite element method, and spectral approaches provide more adaptability in dealing with intricate boundary conditions and geometries. Nevertheless, these

approaches frequently require substantial computer resources and can encounter stability and convergence issues when tackling highly nonlinear or stiff problems (Batchelor, 2000; White, 2006).

The DTM is a highly efficient analytical technique used for solving differential equations. DTM simplifies the solution process of differential equations by converting them into a sequence of algebraic equations, while yet ensuring high accuracy and efficiency. The effectiveness and reliability of this approach have been demonstrated in a range of engineering applications, showcasing its adaptability and resilience (Zhou, 1986; Arikoglu & Ozkol, 2006).

Recent studies underscore the effectiveness and versatility of the DTM in solving complex differential equations (Abdulghafor, 2023; Derikvand et al., 2023). The DTM has emerged as a valuable analytical tool in fluid mechanics, offering an effective approach to solving nonlinear differential equations commonly encountered in this field. Arikoglu and Ozkol (2005) pioneered DTM's use in fluid mechanics by solving the Blasius equation for boundary layer flow across a flat plate. A high precision and less computational time were recorded in their study. Thereafter, the DTM has been the choice several researchers in fluid flow (Ali et al., 2023; Nadeem et al., 2024; Kumar, 2020; Abouzeid & Ibrahim, 2024;

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Wellot, 2022; Zhao et al., 2023; Sadiq Murad and Hamasalh, 2023; Anwar et al., 2022; Kumar, 2020; Nouri, 2013).

The DTM has been a prominent technique for solving differential equations in recent years. Its effectiveness lies in its straightforwardness and precision (Zhou, 1986). The DTM offers a systematic approach to solving differential equations, which is especially valuable when dealing with complex boundary conditions. The integration of DTM with the shooting technique is a highly promising strategy for enhancing the precision and convergence of solutions for nonlinear differential equations.

Conventional analytical techniques frequently encounter difficulties when attempting to solve the nonlinear differential equations that govern these types of flows. In order to tackle these difficulties, numerical and semi-analytical methods have been created. Amongst all these techniques, the shooting method is a widely recognised numerical approach used to solve boundary value problems. This requires the creation of more resilient techniques to improve the precision and effectiveness of solutions.

The shooting method is a potent numerical technique employed for solving boundary value problems. The shooting method is a valuable technique for solving nonlinear differential equations by transforming boundary value problems into initial value problems and iteratively altering the initial conditions to satisfy the boundary conditions (Press et al., 2007; Stoer & Bulirsch, 2002). The efficacy of the shooting method lies in its ability to transform a BVP into an IVP, which can then be solved using well-established methods for IVPs. Recent studies have demonstrated the effectiveness of this method in various complex applications. Notable applications include vibration analysis of beams (Hołubowski and Jarczewska, 2023), Burgers' equation (Gie et al., 2023) and nonlinear two-point boundary value problems (Filipov et al., 2019) where computational efficiency and precision were achieved. Nevertheless, conventional shooting techniques may have difficulties in maintaining stability and achieving convergence, particularly when dealing with rigid or highly nonlinear issues (Ascher et al., 1995).

To overcome these constraints, hybrid approaches, which integrate multiple analytical or numerical techniques, have become increasingly popular for solving complex and nonlinear differential equations. These methods utilise the advantages of specific methodologies, resulting in enhanced precision, effectiveness, and resilience in tackling various mathematical and engineering issues. Several research have adopted this means to tackle various complex problems which are exemplified in Wazwaz (2009), El-Zahar (2013), Boti (2021) and Nayfeh (2011). The entire works highlighted the effectiveness of hybrid methods in handling complex nonlinearities and enhancing solution accuracy and convergence rates.

In this work, the modified shooting method advances the conventional methodology by integrating strategies that boost convergence and stability. When this method is combined with DTM, it provides a strong and effective answer to complex flow problems. The combination of the modified shooting method and DTM efficiently utilises the advantages of both techniques, guaranteeing stability and convergence. This

approach is particularly useful in dealing with the nonlinearity of the governing equations.

The integration of the modified shooting method with the DTM has demonstrated significant efficacy in the analysis of Couette-Poiseuille flow. This hybrid technique offers engineers and researchers a potent tool, allowing them to analyse complex flow problems with enhanced precision and effectiveness. The utilisation of this integrated approach has demonstrated substantial progress in comprehending and simulating fluid dynamics, underscoring its potential for wider implementation across diverse engineering disciplines.

However, the study of Couette-Poiseuille flow continues to be difficult because of the nonlinearity and complexity of the governing equations. Conventional numerical techniques can be computationally demanding and may encounter problems with stability and convergence. Although the DTM provides a simplified and efficient approach for solving differential equations, it may not be enough for very complex flow situations. The enhanced shooting method, an improvement over the standard shooting method, offers increased stability and convergence. However, it still necessitates robust procedures to successfully tackle nonlinearity. Hence, it is imperative to adopt a holistic technique that amalgamates the advantages of the shooting method and the DTM to analyse Couette-Poiseuille flow with enhanced precision and efficacy. The aim of this study is to fill this gap by creating and implementing a hybrid approach to solve the Couette-Poiseuille flow problem in a more efficient manner.

This work provides an analysis of Couette-Poiseuille flow using a modified shooting approach that is combined with the differential transform approach, DTM. The utilisation of the modified shooting approach improves the convergence of the numerical solution by exploiting the series expansion characteristics of DTM, resulting in a more efficient and precise solution for the flow equations. This integrated approach addresses the constraints of conventional methods and offer a more all-encompassing solution to the Couette-Poiseuille flow problem.

This study highlights the promise of the proposed technique as a robust tool for solving fluid mechanics problems. It also suggests that this methodology can be deployed to a wider range of engineering challenges, thereby contributing to the development of computational approach in fluid dynamics.

The organisation of this document is as follows: Section II presents the mathematical representation of Couette-Poiseuille flow together with the algorithm of the modified shooting approach and its incorporation with DTM. Section III depicts the numerical implementation of the proposed algorithm. Section IV provides an overview of the findings and subsequent discussions, while Section V delivers the conclusions.

II. METHODOLOGY

A Couette-Poiseuille flow refers to the flow of fluid that is confined between two walls. In this scenario, the upper wall is moving at a predetermined velocity, V_0 , while the distance between the upper wall and the bottom wall is $Y = L$. In

addition, the flow is influenced by a predetermined pressure gradient, $\partial R/\partial X$. g is the gravitational acceleration. Figure 1 shows the Couette flow that is being driven by a pressure gradient and a moving upper wall, then stationary at the lower wall.

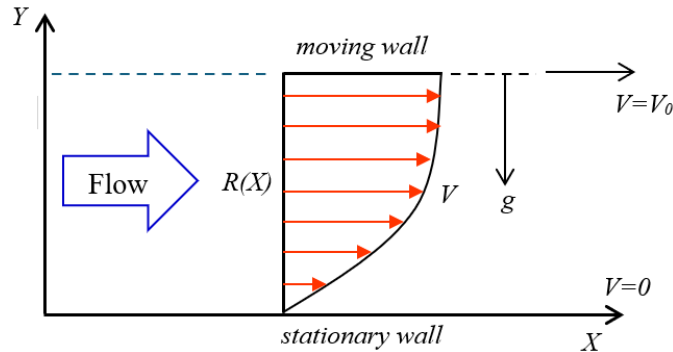


Figure 1: Pressure gradient-driven Couette flow with a moving top wall

Assuming that the vertical velocity component V is equal to zero, we may deduce from the principle of continuity that the partial derivative of the velocity component V with respect to the streamwise direction X is equal to zero. This suggests that the velocity V is solely dependent on the crosswise direction Y . In other words, the velocity in the X -direction is only influenced by changes in the Y -direction.

The equation of motion in the Y -direction can be simplified as $\partial R/\partial X = -\rho g$, while the equation of motion in the X -direction takes the following form:

$$\rho V \left(\frac{\partial V}{\partial X} \right) = \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) \mu - \frac{\partial R}{\partial X} \quad (1)$$

This, as a result of the aforementioned assumptions and consequences, simplifies to

$$\frac{\partial^2 V}{\partial Y^2} = \frac{1}{\mu} \frac{dR}{dX} \quad (2)$$

Under the no-slip boundary conditions, the velocity V at the starting point is zero, $V(0) = 0$, and at the end point it is equal to $V(L)$, $V(L) = V_0$. In order to express Eq. (2) in a more suitable and generic manner, we introduce dimensionless variables:

$$v = \frac{V}{V_0}, y = \frac{Y}{L}, G = -\frac{dR}{dX} \frac{L^2}{V_0 \mu} \quad (3)$$

The velocity profile $v(y)$ in a channel with two parallel plates can now be described by the following dimensionless form,

$$\frac{\partial^2 v}{\partial y^2} = -G \quad (4)$$

Where G is the dimensionless pressure gradient.

A. Hybrid Shooting-Differential Transform Method (HSDTM)

The shooting approach was implemented to tackle second order boundary value problems. This method involves solving second order boundary value problems by combining two initial value problems. It is observed that the differential transform method yields a rapidly converging series solution for solving initial value problems. This motivates us to utilise this solution in solving second order linear boundary value

problems through the linear shooting method. Let us examine the second order boundary value problem.

$$w''(x) = p(x)w'(x) + q(x)w(x) + r(x) \quad (5)$$

for $a \leq x \leq b$, subject to $w(a) = \alpha$ and $w(b) = \beta$. If it is assumed that Eq. (5) possesses a singular solution, the shooting method can be employed to determine an approximate unique solution. This involves separating the problem into two separate initial value problems to give,

$$z''(x) = p(x)z'(x) + q(x)z(x) + r(x) \quad (6)$$

Subject to $z(a) = \alpha; z'(a) = 0$

$$\text{and } m''(x) = p(x)m'(x) + q(x)m(x) \quad (7)$$

Subject to $m(a) = 0; m'(a) = 1$

The solutions of Equations (6) and (7) are derived by implementing the DTM on them. Let $z(x)$ and $m(x)$ denote the solutions respectively. Then the linear shooting method solution becomes,

$$w(x) = z(x) + \frac{\beta - z(b)}{m(b)} m(x) \quad (8)$$

where $m(b) \neq 0$

B. Implementation Algorithm

Consider the second-order boundary value problem,

$$y''(x) = f(x, y, y') \quad (9)$$

with boundary conditions:

$$y(a) = \alpha \text{ and } y(b) = \beta \quad (10)$$

C. The shooting method for boundary value problems

i. Convert the Second-Order ODE to a System of First-Order ODEs.

- Then, the original second-order ODE can be written as a system of first-order ODEs:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= f(x, y_1, y_2) \end{aligned} \quad (11)$$

ii. Initial Guesses for the Slope

- Make an initial guess for the value of $y'(a)$. Let us denote this initial guess by

$$y_2(a) = s \quad (12)$$

iii. Solve the Initial Value Problem (IVP)

- With the initial conditions:

$$y_1(a) = \alpha \text{ and } y_2(a) = s \quad (13)$$

- Solve the IVP using a suitable numerical method (e.g., Runge-Kutta method) to find $y_2(b)$.

iv. Adjust the Slope Using a Root-Finding Method:

- Define a function $\phi(s) = y_1(b) - \beta$. The goal is to find the value of s such that $\phi(s) = 0$. This can be done using a root-finding method like the Newton-Raphson method or the secant method.

Secant Method:

- Make two initial guesses s_0 and s_1
 - Compute $\phi(s_0)$ and $\phi(s_1)$ by solving the IVP
 - Update the guess using:
- $$S_{k+1} = S_k - \phi(s_k) \frac{S_k - S_{k+1}}{\phi(s_k) - \phi(s_{k-1})} \quad (14)$$
- Repeat until $|\phi(s_k)|$ is sufficiently small.

Newton-Raphson Method:

- Make two initial guesses s_0
- Compute $\phi(s_0)$ by solving the IVP
- Update the guess using:

$$S_{k+1} = S_k - \frac{\phi(S_k)}{\phi'(S_k)} \tag{15}$$

Where $\phi'(S_k)$ can be approximated using a finite difference:

$$\phi'(S_k) \approx \frac{\phi(S_k + \varepsilon) - \phi(S_k)}{\varepsilon} \tag{16}$$

- Repeat until $|\phi(S_k)|$ is sufficiently small.

v. Solution

Once S is found such that $\phi(s) = 0$, solve the IVP with $y_1(a) = \alpha$ and $y_2(a) = s$ to obtain the solution $y(x)$ for the original BVP.

D. Hybrid shooting-differential transform method boundary value problems:

- i. Repeat steps (1) and (2) in section C.1
- ii. Differential Transform Method

Apply the DTM to each IVP:

Initialize the differential transform of y_1

$$Y_i(k) = \frac{1}{k!} \left. \frac{d^k y_i(x)}{dx^k} \right|_{x=a} \tag{17}$$

Derive the transformed equations using the DTM rules for both IVPs.

- iii. Deploy the linear shooting method, Eq. (8) to combine the two solutions in step (2).

III. NUMERICAL EXAMPLE

The primary focus of this numerical experiment is to analyse Couette-Poiseuille flow using a modified shooting method integrated with the Differential Transform Method. The goal is to demonstrate the effectiveness of this hybrid approach in solving the nonlinear differential equations governing the flow and to compare the results with existing analytical and numerical solutions. For the purpose of this illustration, a linearized version of the Couette-Poiseuille flow will be used. The governing equations for Couette-Poiseuille flow are derived from the Navier-Stokes equations under the assumptions of steady, incompressible, and laminar flow. The velocity profile $v(y)$ is described by Eq. (4) with the following boundary conditions:

$$v(0) = 2 \text{ and } v(3) = U \tag{18}$$

where U is the dimensionless velocity of the moving plate.

The solution of Eq. (4) is sought through the hybrid shooting-differential transform method with value of $U = 7$. The solution is determined for various values of the dimensionless pressure gradient G ; namely $G = -8, -4, 0, 4$ and 8 . The velocity profiles for the Couette-Poiseuille flow with various pressure gradients is simulated using the conventional shooting method via fourth-order Runge-Kutta and Secant iteration methods. Finally, the modified shooting method with DTM is also simulated. These results are

compared with the exact solution of the problem, $\frac{1}{6}(-3Px^2 + 9Px + 10x + 12)$.

IV. RESULTS AND DISCUSSION

The results of the solution from the conventional shooting method and that from the hybrid approach are shown and discussed. Figure 2 depicts the velocity profile $v(y)$ of the first and second guesses, s_0 and s_1 for different values of the dimensionless pressure gradient G using the shooting method together with the boundary points.

As expected, the results diverged away from the boundary points for all the values of the dimensionless pressure gradient G . The velocity profile $v(y)$ of the updated guess, s^* for different values of the dimensionless pressure gradient G using the shooting method is shown in Figure 3 with the converged results. The results converged within the boundary points at the third iteration.

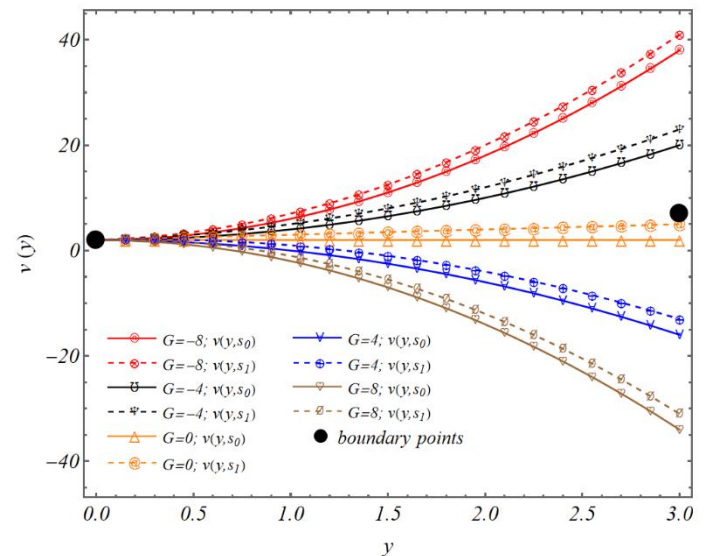


Figure 2: The velocity profile $v(y)$ of the first and second guesses, s_0 and s_1 for different values of the dimensionless pressure gradient G using the shooting method

Figure 4 illustrates the velocity profile $v(y)$ for various values of the dimensionless pressure gradient G . The velocity profiles are computed using three approaches: analytical, shooting and hybrid shooting-differential transform method. The result show that the shooting method and the HSDTM converged with the exact solution. However, the HSDTM has shown a supreme convergence and computational time compared with the shooting method. The curves represent different values of G , specifically, $-8 \leq G \leq 8$. The velocity profiles are parabolic in nature, typical of fluid flow between two parallel plates (Poiseuille flow). Symmetry is evident about the y -axis at $y = 1.5$, which might indicate the centre line of the channel. For positive values of G (e.g., $G = 4$ and $G = 8$), the velocity profiles are upward concave. Higher values of G result in higher peak velocities, indicating an increased driving force for the fluid flow. For negative values

of G (e.g., $G = -4$ and $G = -8$), the velocity profiles are downward concave. Higher negative values of G result in lower (more negative) peak velocities, indicating a reverse driving force for the fluid flow (White, 1991). When $G = 0$, the velocity profile is a straight line passing through the origin. This suggests that there is no driving force, and the fluid remains stationary.

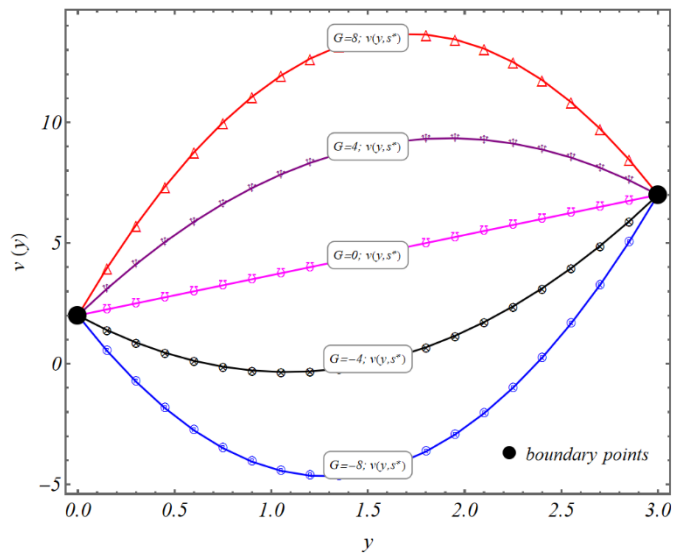


Figure 3: The velocity profile $v(y)$ of the updated guess, s^* for different values of the dimensionless pressure gradient G using the shooting method. This is the converged result

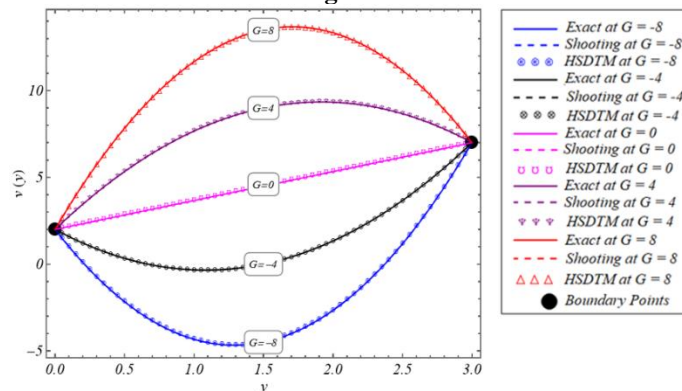


Figure 4: The velocity profile $v(y)$ for different values of the dimensionless pressure gradient G using the Shooting Method, HSDTM and Exact Solution

The HSDTM results align well with the analytical solutions for simple Poiseuille flow. For example, for a positive pressure gradient in a Newtonian fluid, the velocity profile follows a parabolic distribution which is well captured by the method (Bejan, 2013).

The shear stress at the walls can be inferred from the slope of the velocity profile. Higher absolute values of G result in steeper slopes at the walls, indicating higher shear stress (Shah and London, 1978). Understanding the velocity profiles under different pressure gradients is crucial for the design of fluid transport systems, such as pipelines and microfluidic devices (Chhabra and Richardson, 2011). The results are also relevant

for the analysis of blood flow in arteries, where varying pressure gradients can have significant physiological impacts. The HSDTM has been validated against other numerical methods and exact solution, showing high accuracy and efficiency in solving boundary value problems in fluid dynamics. The HSDTM demonstrated improved convergence rates and accuracy compared to the traditional shooting method. The hybrid approach required fewer iterations and computational resources, highlighting its efficiency in solving nonlinear differential equations.

The results indicate that the HSDTM is a robust and efficient approach for solving Couette-Poiseuille flow problems. The improved convergence and accuracy make this method particularly suitable for complex fluid dynamics problems where traditional methods may struggle. The success of this hybrid method suggests potential applications in other areas of fluid mechanics and engineering, where similar boundary value problems arise. Future work could extend this approach to three-dimensional flow problems and explore its application to other types of nonlinear differential equations.

V. CONCLUSION

This work presents a successful implementation of a hybrid approach that combines the modified shooting method with the differential transform method for the analysis of Couette-Poiseuille flow. This hybrid approach efficiently overcame the constraints of conventional numerical methods. From the findings of the study, we have concluded that:

- The HSDTM not only achieves a high level of accuracy in solving the governing equations of Couette-Poiseuille flow, but also greatly enhances the reliability of the solution process.
- The progress of numerical approach in fluid dynamics, providing a more efficient solution framework for many engineering applications.
- The HSDTM exhibits significant promise for wider applications, hence facilitating additional investigation and advancement in diverse fluid mechanics problems.

Subsequent research can broaden the application of this hybrid approach to more intricate flow situations, hence augmenting its usefulness and influence in the realm of fluid mechanics.

AUTHOR CONTRIBUTIONS

T. T. Akano: Conceptualization, Methodology, Validation, Writing – original draft. **O. S. Motsamai:** Writing – review & editing. **H. O. Onovo:** Writing – review & editing.

REFERENCES

Abouzeid, M., and Ibrahim, M. (2024). Multi-step differential transform method for both Hall currents and mixed convection effects on MHD flow of non-Newtonian fluid with Al₂O₃ nanoparticles. *Egyptian Journal of Chemistry*, 67 (6): 225-232.

Ali, K.; A. A. Faridi; N. Khan; K. S. Nisar and S. Ahmad. (2023). On the suitability of differential transform method for solving the self-similar channel flow problems.

ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 103(1), e202100358.

Al-Rozbayani, A. M. and Shammar, A. H. (2023, February). Applied reduced differential transform method to the Wu-Zhang equation in (1+1)-dimensional. In *American Institute of Physics Conference Series*, 2414 (1): 040004. <https://doi.org/10.1063/5.0114833>.

Anwar, M. I.; H. Firdous; A. A. Zubaidi; N. Abbas and S. Nadeem. (2022). Computational analysis of induced magnetohydrodynamic non-Newtonian nanofluid flow over nonlinear stretching sheet. *Progress in Reaction Kinetics and Mechanism*, 47, 14686783211072712.

Arikoglu, A. and Ozkol, I. (2005). Inner-outer matching solution of Blasius equation by DTM. *Aircraft Engineering and Aerospace Technology*, 77 (4): 298-301.

Arikoglu, A. and Ozkol, I. (2006). Solution of differential-difference equations by using differential transform method. *Applied Mathematics and Computation*,

Ascher, U. M.; R. M. Mattheij and R. D. Russell. (1995). Numerical solution of boundary value problems for ordinary differential equations. *SIAM*.

Batchelor, G. K. (2000). An introduction to fluid dynamics. *Cambridge University Press*.

Botti, M.; D. C. Quiroz; D. A. Di Pietro and A. Harnist. (2021). A hybrid high-order method for creeping flows of non-Newtonian fluids. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55 (5): 2045-2073.

Derikvand, M.; F. Farhatnia and D. H. Hodges. (2023). Functionally graded thick sandwich beams with porous core: Buckling analysis via differential transform method. *Mechanics Based Design of Structures and Machines*, 51 (7): 3650-3677. <https://doi.org/10.1080/15397734.2021.1931309>

El-Zahar, E. R. (2013). Approximate analytical solutions of singularly perturbed fourth order boundary value problems using differential transform method. *Journal of King Saud University-Science*, 25 (3): 257-265.

Filipov, S. M.; I. D. Gospodinov and I. Faragó (2019). Replacing the finite difference methods for nonlinear two-point boundary value problems by successive application of the linear shooting method. *Journal of Computational and Applied Mathematics*, 358: 46-60.

Gie, G. M.; C. Y. Jung and Lee, H. (2023). Semi-analytic shooting methods for Burgers' equation. *Journal of Computational and Applied Mathematics*, 418, 114694.

natural vibrations of non-prismatic multi-segment beams. *Applied Mathematics and Mechanics*, 44 (12): 2243-2252.

Kumar, M. (2020). Study of differential transform technique for transient hydromagnetic Jeffrey fluid flow from a stretching sheet. *Nonlinear Engineering*, 9 (1): 145-155.

Kumar, M.; G. J. Reddy; N. N. Kumar and O. A. Bég (2019). Application of differential transform method to unsteady free convective heat transfer of a couple stress fluid over a stretching sheet. *Heat Transfer—Asian Research*, 48 (2): 582-600.

Nayfeh, A. H. (2011). Introduction to perturbation techniques. *John Wiley & Sons*.

Nadeem, S.; B. Ishtiaq; J. Alzabut and S. M. Eldin. (2024). Implementation of differential transform method on the squeezing flow of trigonometric non-Newtonian fluid between two heated plates. *International Journal of Modern Physics B*, 38 (24): 2450326.

Nouri, R.; D. D. Ganji and M. Hatami. (2013). MHD nanofluid flow analysis in a semi-porous channel by a combined series solution method. *Challenges in Nano and Micro Scale Science and Technology*, 1 (2): 124-137.

Press, W. H.; S. A. Teukolsky; W. T. Vetterling and B. P. Flannery. (2007). Numerical recipes: The art of scientific computing. *Cambridge University Press*.

Sadiq Murad, M. A. and Hamasalh, F. K. (2023). Numerical study for fractional-order magnetohydrodynamic boundary layer fluid flow over stretching sheet. *Punjab University Journal of Mathematics*, 55 (2).

Stoer, J. and Bulirsch, R. (2002). Introduction to numerical analysis. *Springer*.

Wazwaz, A. M. (2009). The variational iteration method for analytic treatment for linear and nonlinear ODEs. *Applied Mathematics and Computation*, 212 (1): 120-134.

Wellot, Y. A. S. (2022). Application of the Reduced Differential Transform Method to Solve the Navier-Stokes Equations. *Pure and Applied Mathematics Journal*, 7 (1): 96-101.

White, F. M. (2006). Viscous fluid flow. *McGraw-Hill*.

Zhao, Q.; W. Liu; W. Yu, and F. Cai. (2023). Dynamics of a fluid-conveying pipe by a hybrid method combining differential transformation and Galerkin discretization. *Iranian Journal of Science and Technology, Transactions of Mechanical Engineering*, 1-13.

Zhou, J. K. (1986). Differential transformation and its applications for electrical circuits. *Huarjung University Press*.

Holubowski, R. and Jarczewska, K. (2023). A two-parameter multiple shooting method and its application to the